A mathematical model will be linear if the differential equations describing the system has constant coefficients (or the coefficients may be functions of independent variables). If the coefficients of the differential equation describing the system are constants then the model is ***linear time invariant.*** If the coefficients of differential equations governing the system are functions of time then the model is ***linear time varying.***

The differential equations of a linear time invariant system can be reshaped into different form for the convenience of analysis. One such model for single input and single output system analysis is transfer function of the system. The ***transfer function*** of a system is defined as the ratio of Laplace transform of output to the Laplace transform of input with zero initial conditions.

$$\text{Transfer function} = \left.\frac{\text{Laplace Transform of output}}{\text{Laplace Transform of input}}\right|_{\text{with zero initial condition}} \quad .....(1.1)$$

The transfer function can be obtained by taking Laplace transform of the differential equations governing the system with zero initial conditions and rearranging the resulting algebraic equations to get the ratio of output to input.

1.4 MECHANICAL TRANSLATIONAL SYSTEMS

The model of mechanical translational systems can be obtained by using three basic elements ***mass, spring and dash-pot.*** These three elements represents three essential phenomena which occur in various ways in mechanical systems.

The weight of the mechanical system is represented by the element ***mass*** and it is assumed to be concentrated at the center of the body. The elastic deformation of the body can be represented by a ***spring.*** The friction existing in rotating mechanical system can be represented by the ***dash-pot.*** The dash-pot is a piston moving inside a cylinder filled with viscous fluid.

When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The force acting on a mechanical body are governed by ***Newton's second law of motion.*** For translational systems it states that the sum of forces acting on a body is zero. (or Newton's second law states that the sum of applied forces is equal to the sum of opposing forces on a body).

LIST OF SYMBOLS USED IN MECHANICAL TRANSLATIONAL SYSTEM

x = Displacement, m

$v = \frac{dx}{dt}$ = Velocity, m/sec

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ = Acceleration, m/sec^2

f = Applied force, N (Newtons)

f_m = Opposing force offered by mass of the body, N

f_k = Opposing force offered by the elasticity of the body (spring), N

f_b = Opposing force offered by the friction of the body (dash - pot), N

M = Mass, kg

K = Stiffness of spring, N/m

B = Viscous friction co-efficient, N-sec/m

Note : *Lower case letters are functions of time*

$$\therefore\ M_2\frac{d^2y_2}{dt^2}+K_2(y_2-y_1)=0$$

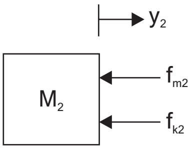

Fig 3.

On taking Laplace transform of above equation we get,

$$M_2s^2Y_2(s)+K_2[Y_2(s)-Y_1(s)]=0$$

$$Y_2(s)\,[M_2s^2+K_2]-Y_1(s)\,K_2=0$$

$$\therefore\ Y_1(s)=Y_2(s)\,\frac{M_2s^2+K_2}{K_2} \qquad .....(3)$$

Substituting for $Y_1(s)$ from equation (3) in equation (2) we get,

$$Y_2(s)\left[\frac{M_2s^2+K_2}{K_2}\right][M_1s^2+Bs+(K_1+K_2)]-Y_2(s)\,K_2=F(s)$$

$$Y_2(s)\left[\frac{(M_2s^2+K_2)\,[M_1s^2+Bs+(K_1+K_2)]-K_2^2}{K_2}\right]=F(s)$$

$$\therefore\ \frac{Y_2(s)}{F(s)}=\left[\frac{K_2}{[M_1s^2+Bs+(K_1+K_2)]\,(M_2s^2+K_2)-K_2^2}\right]$$

RESULT

The differential equations governing the system are,

1. $M_1\dfrac{d^2y_1}{dt^2}+B\,\dfrac{dy_1}{dt}+K_1(y_1-y_2)=f(t)$

2. $M_2\dfrac{d^2y_2}{dt^2}+K_2(y_2-y_1)=0$

The transfer function of the system is,

$$\frac{Y_2(s)}{F(s)}=\frac{K_2}{[M_1s^2+Bs+(K_1+K_2)]\,[M_2s^2+K_2]-K_2^2}$$

EXAMPLE 1.3

Determine the transfer function, $\dfrac{X_1(s)}{F(s)}$ and $\dfrac{X_2(s)}{F(s)}$ for the system shown in fig 1.

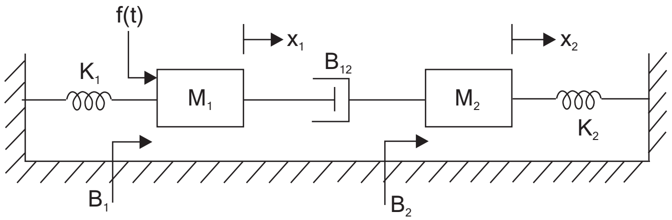

Fig 1.

SOLUTION

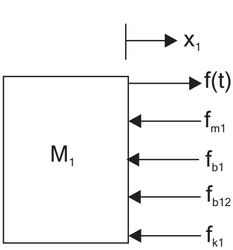

Fig 2.

Let, Laplace transform of f(t) = $\mathcal{L}\{f(t)\}$ = F(s)

Laplace transform of x_1 = $\mathcal{L}\{x_1\}$ = $X_1(s)$

Laplace transform of x_2 = $\mathcal{L}\{x_2\}$ = $X_2(s)$

Consider an ideal elastic element, torsional spring as shown in fig 1.17, which has negligible moment of inertia and friction. Let a torque be applied on it. The torsional spring will offer an opposing torque which is proportional to angular displacement of the body.

Let, T = Applied torque.

T_k = Opposing torque due to elasticity.

$T_k \propto \theta$ or $T_k = K\theta$

By Newton's second law, $\boxed{T = T_k = K\theta}$ .....(1.10)

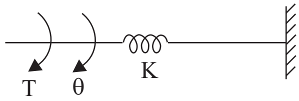

***Fig 1.17 :** Ideal spring with one end fixed to reference.*

When the spring has angular displacement at both ends as shown in fig 1.18 the opposing torque is proportional to difference between angular displacement at both ends.

$T_k \propto (\theta_1 - \theta_2)$ or $T_k = K(\theta_1 - \theta_2)$

$\boxed{\therefore T = T_k = K(\theta_1 - \theta_2)}$ .....(1.11)

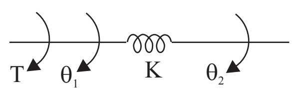

***Fig 1.18 :** Ideal spring with angular displacement at both ends.*

Guidelines to determine the Transfer Function of Mechanical Rotational System

1. In mechanical rotational system, the differential equations governing the system are obtained by writing torque balance equations at nodes in the system. The nodes are meeting point of elements. Generally the nodes are mass elements with moment of inertia in the system. In some cases the nodes may be without mass element.
2. The angular displacement of the moment of inertia of the masses (nodes) are assumed as θ_1, θ_2, θ_3, etc., and assign a displacement to each mass (node). The first derivative of angular displacement is angular velocity and the second derivative of the angular displacement is angular acceleration.
3. Draw the free body diagrams of the system. The free body diagram is obtained by drawing each moment of inertia of mass separately and then marking all the torques acting on that body. Always the opposing torques acts in a direction opposite to applied torque.
4. The mass has to rotate in the direction of the applied torque. Hence the angular displacement, velocity and acceleration of the mass will be in the direction of the applied torque. If there is no applied torque then the angular displacement, velocity and acceleration of the mass is in a direction opposite to that of opposing torque.
5. For each free body diagram write one differential equation by equating the sum of applied torques to the sum of opposing torques.
6. Take Laplace transform of differential equation to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the relation between output variable and input variable. This ratio is the transfer function of the system.

Note :

Laplace transform of $\theta = \mathcal{L}\{\theta\} = \theta(s)$

Laplace transform of $\dfrac{d\theta}{dt} = \mathcal{L}\left\{\dfrac{d\theta}{dt}\right\} = s\,\theta(s)$ *(with zero initial conditions)*

Laplace transform of $\dfrac{d^2\theta}{dt^2} = \mathcal{L}\left\{\dfrac{d^2\theta}{dt^2}\right\} = s\,\theta(s)$ *(with zero initial conditions)*

Taking Laplace transform of the above equations with zero initial conditions we get,

$$I_a(s)\, R_a + L_a s I_a(s) + E_b(s) = V_a(s) \qquad .....(1.16)$$

$$T(s) = K_t I_a(s) \qquad .....(1.17)$$

$$Js^2\theta(s) + B\, s\, \theta(s) = T(s) \qquad .....(1.18)$$

$$E_b(s) = K_b s\theta\,(s) \qquad .....(1.19)$$

On equating equations (1.17) and (1.18) we get,

$$K_t I_a(s) = (Js^2 + Bs)\,\theta(s)$$

$$I_a(s) = \frac{(Js^2 + Bs)}{K_t}\,\theta(s) \qquad .....(1.20)$$

Equation (1.16) can be written as,

$$(R_a + sL_a)\, I_a(s) + E_b(s) = V_a(s) \qquad .....(1.21)$$

Substituting for $E_b(s)$ and $I_a(s)$ from equation (1.19) and (1.20) respectively in equation (1.21),

$$(R_a + sL_a)\,\frac{(Js^2 + Bs)}{K_t}\,\theta(s) + K_b s\theta(s) = V_a(s)$$

$$\left[\frac{(R_a + sL_a)(Js^2 + Bs) + K_b K_t s}{K_t}\right]\theta(s) = V_a(s)$$

The required transfer function is $\dfrac{\theta(s)}{V_a(s)}$

$$\therefore\ \frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)(Js^2 + Bs) + K_b K_t s} \qquad ....(1.22)$$

$$= \frac{K_t}{R_a Js^2 + R_a Bs + L_a Js^3 + L_a Bs^2 + K_b K_t s}$$

$$= \frac{K_t}{s[\, JL_a s^2 + (JR_a + BL_a)s + (BR_a + K_b K_t)\,]}$$

$$= \frac{K_t/JL_a}{s\left[s^2 + \left(\dfrac{JR_a + BL_a}{JL_a}\right)s + \left(\dfrac{BR_a + K_b K_t}{JL_a}\right)\right]} \qquad .....(1.23)$$

The transfer function of armature controlled dc motor can be expressed in another standard form as shown below. From equation (1.22) we get,

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)(Js^2 + Bs) + K_b K_t s} = \frac{K_t}{R_a\left(\dfrac{sL_a}{R_a} + 1\right)Bs\left(1 + \dfrac{Js^2}{Bs}\right) + K_b K_t s}$$

$$= \frac{K_t/R_a B}{s\left[(1 + sT_a)(1 + sT_m) + \dfrac{K_b K_t}{R_a B}\right]} \qquad .....(1.24)$$