then
$$A+B=\begin{bmatrix}1&2&3\\5&1&4\\7&8&9\end{bmatrix}+\begin{bmatrix}1&3&5\\5&0&1\\3&2&12\end{bmatrix}$$
$$=\begin{bmatrix}1+1&2+3&3+5\\5+5&1+0&4+1\\7+3&8+2&9+12\end{bmatrix}=\begin{bmatrix}2&5&8\\10&1&5\\10&10&21\end{bmatrix}$$

☛ **REMARK**

- If the orders of the matrices are different, then they are not conformable for addition.

### 1.4.2 Substraction of Matrices

Suppose $A$ and $B$ are two matrices of same order, then the substraction of $A$ and $B$, *i.e.*, $A–B$ is obtained by substracting each element of $B$ from the corresponding element of $A$. If $A$ and $B$ are of order $m \times n$, then $A-B$ will be of order $m \times n$.

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$

then $A-B = [a_{ij} - b_{ij}]_{m\times n}$

**For example:** If
$$A=\begin{bmatrix}1&2&3\\3&4&5\\5&6&7\end{bmatrix} \text{ and } B=\begin{bmatrix}0&5&2\\3&-2&2\\5&7&8\end{bmatrix}$$

then
$$A-B=\begin{bmatrix}1&2&3\\3&4&5\\5&6&7\end{bmatrix}-\begin{bmatrix}0&5&2\\3&-2&2\\5&7&8\end{bmatrix}$$
$$=\begin{bmatrix}1-0&2-5&3-2\\3-3&4-(-2)&5-2\\5-5&6-7&7-8\end{bmatrix}=\begin{bmatrix}1&-3&1\\0&6&3\\0&-1&-1\end{bmatrix}$$

☛ **REMARK**

- If the order of matrices are different, then they are not conformable for substraction.

### 1.4.3 Multiplication of a Matrix by a Scalar

Suppose $A$ is a matrix of order $m \times n$ and $k$ is a scalar, then the multiplication of $A$ by $k$, *i.e.* $kA$ is obtained by multiplying each element of $A$ by $k$.

Let $A = [a_{ij}]_{m\times n}$ $\forall$ $1 \le i \le m$ and $1 \le j \le n$, then $kA = [ka_{ij}]_{m\times n}$

**For example :** If
$$A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} \text{ and } k = 3,$$

then
$$3A=3\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$$
$$=\begin{bmatrix}3\times1&3\times2&3\times3\\3\times4&3\times5&3\times6\\3\times7&3\times8&3\times9\end{bmatrix}=\begin{bmatrix}3&6&9\\12&15&18\\21&24&27\end{bmatrix}$$

$$= a[a_{ij}]_{m\times n} + b[a_{ij}]_{m\times n}$$
$$= aA + bA$$

Hence $\qquad (a + b)A = aA + bA$

(iii) *If $A$ is a matrix of order $m \times n$ and $a$, $b$ are two scalars, then $a(bA) = (ab)A$*

**Proof.** Let $A = [a_{ij}]_{m\times n}$, then

$$\begin{aligned} a(bA) &= a(b[a_{ij}]_{m\times n}) \\ &= [a(ba_{ij})]_{m\times n} && \text{(By scalar multiplication)} \\ &= [(ab)a_{ij}]_{m\times n} && (\because \text{ Numbers are associative}) \\ &= (ab)[a_{ij}]_{m\times n} \\ &= (ab)\,A \end{aligned}$$

Hence $\qquad a(bA) = (ab)A.$

(iv) If $A$ is a matrix of order $m \times n$ and $k$ is any scalar, then $(-k)A = -(kA) = k(-A)$

**Proof.** Let $A = [a_{ij}]_{m\times n}$, then

$$\begin{aligned} (-k)A &= (-k)[a_{ij}]_{m\times n} \\ &= [(-k)a_{ij}]_{m\times n} && \text{(By scalar multiplication)} \\ &= [-ka_{ij}]_{m\times n} \\ &= -[ka_{ij}]_{m\times n} \\ &= -(kA) \end{aligned}$$

Now
$$\begin{aligned} (-k)A &= (-k)[a_{ij}]_{m\times n} \\ &= [(-k)a_{ij}]_{m\times n} \\ &= [k(-a_{ij})]_{m\times n} \\ &= k[-a_{ij}]_{m\times n} \\ &= k(-A) \end{aligned}$$

Hence $\qquad (-k)A = -(kA) = k(-A).$

## 1.7 MULTIPLICATION OF MATRICES

Let $A$ and $B$ be two matrices of order $m \times n$ and $n \times p$ respectively. Then a matrix $C$ of order $m \times p$ is obtained by multiplying each row of $A$ to each column of $B$.

Suppose $A = [a_{ij}]_{m\times n}$, $B = [b_{jk}]_{n\times p}$, then $C = [c_{ik}]_{m\times p}$ is known as the multiplication of $A$ and $B$ if

$$c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$$

and hence we can write $\qquad C = AB$

second order determinant, which is called the minor of the element $a_{ij}$. It is denoted by $M_{ij}$ . Therefore, in a determinant of order 3, we may get 9 minors corresponding to the 9 elements of the determinant.

For example, in determinant (1)

$$\text{Minor of } a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$$

and

$$\text{Minor of } a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = M_{32}$$

If we expand the determinant along the first row, then

$$\Delta = (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} + (-1)^{1+3} a_{13} M_{13}$$
$$= a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

Similarly, along second column, we can write

$$\Delta = -a_{12} M_{12} + a_{22} M_{22} - a_{32} M_{32}$$

### 1.11.2 Cofactor

If we multiply the minor $M_{ij}$ by $(-1)^{i+j}$ . Then resulting value is called cofactor of the element $a_{ij}$. If $A_{ij}$ is the cofactor of $a_{ij}$, then we write

$$\text{Cofactor of } a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$$

$$\text{Cofactor of } a_{21} = A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{Cofactor of } a_{32} = A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Hence, cofactor of $a_{ij}$ = $(-1)^{i+j}$ determinant obtained by leaving row and column passing through that element. Therefore, we can write

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$
$$\Delta = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

and

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$$
$$a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$$

## 1.12 SINGULAR AND NON-SINGULAR MATRIX

**Definition.** *A matrix whose determinant value is zero, is said to be singular matrix.*

If the matrix is not singular, then it is said to be non-singular.

**For example :** If $A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$, then its determinant value.

$$|A| = \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 2 \times 9 - 3 \times 6 = 18 - 18 = 0$$

Thus the matrix $A$ is singular.

## 1.13 TRANSPOSE OF A MATRIX

Consider a matrix $A = [a_{ij}]_{m \times n}$. Then a matrix which is obtained by interchanging the rows and columns of $A$ is called the transpose of $A$. It is denoted by $A'$ or $A^T$.

Applying $R_2 \to R_2 - 3R_1$, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} A$$

Again applying $R_1 \to R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} A$$

$\Rightarrow$ $$I_2 = BA$$

$\Rightarrow$ $$A^{-1} = B = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}.$$

(ii) We write

$$A = I_2 A$$

or $$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

Applying $R_2 \to -\frac{1}{5} R_2$, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2/5 & -1/5 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix} A$$

$\Rightarrow$ $$I_2 = BA$$

$\Rightarrow$ $$A^{-1} = B = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}$$

**EXAMPLE 2.** *Find the inverse of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

*by using elementary row-transformation.*

**SOLUTION.** We write $$A = I_3 A$$

or $$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

**For example:**

1. A line $a_1x_1 + a_2x_2 = b$ represents a line in two dimensions which may be considered as a set of these points $(x_1, x_2)$. Therefore, set of points can be written as

$$S = \{(x_1, x_2) : a_1x_1 + a_2x_2 = b\}$$

2. If we consider the set of points lying inside a circle of unit radius with centre at the origin in two dimensional space. Clearly, the points $(x_1, x_2)$ of this set satisfy the inequality

$$x_1^2 + x_2^2 < 1$$

Therefore, set of points is given by

$$S = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$$

### 1.21.2 Hypersphere

In $n$-dimensional space, a hypersphere, with centre $a$ and radius $r(>0)$ is the set of points

$$X = \{x : |x - a| = r\}.$$

The equation of hypersphere in $E^n$ (or $R^n$) is given by $\Sigma(x_i - a_i)^2 = r^2$

☛ **Remarks**

- The set of points inside the hypersphere is the set $X = \{x : |x - a| < r\}$
- The set of points lying inside the hypersphere with centre $a$ and radius $\in > 0$ is said to be $\in$-neighbourhood about the point $a$.

### 1.21.3 Lines and Line Segments

Let $x_1, x_2$ be two distinct points in $n$-dimensional space $E^n$, then the line through the points $x_1$ and $x_2$ is defined to be the set of points given by

$$X = \{x : x = \lambda x_1 + (1 - \lambda)x_2, \text{ for } \lambda \in \boldsymbol{R}\}$$

and the line segment joining two points $x_1$ and $x_2$ in $E^n$ is defined to be the set of points given by

$$X = \{x : x = \lambda x_1 + (1 - \lambda)x_2, 0 \leq \lambda \leq 1\}$$

### 1.21.4 Hyperplane

It is defined as the set of points $(x_1, x_2, \ldots x_n)$ satisfying

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n = z, \text{ (not all } c_i = 0)$$

for prescribed values of $c_1, c_2, \ldots, c_n$ and $z$

We clearly observe that a hyperplane divides the whole space into three mutually disjoint sets as given below:

$$X_1 = \{x : \mathbf{cx} > \boldsymbol{z}\} = \{(x_1, x_2, \ldots, x_n) : c_1x_1 + c_2x_2 + \ldots + c_nx_n > z\}$$
$$X_2 = \{x : \mathbf{cx} = \boldsymbol{z}\} = \{(x_1, x_2, \ldots, x_n) : c_1x_1 + c_2x_2 + \ldots + c_nx_n = z\}$$
$$X_3 = \{x : \mathbf{cx} < \boldsymbol{z}\} = \{(x_1, x_2, \ldots, x_n) : c_1x_1 + c_2x_2 + \ldots + c_nx_n < z\}$$

Here the set $X_1$ and $X_3$ are known as open-half spaces and the sets $\{x : \mathbf{cx} \leq \boldsymbol{z}\}$ and $\{x : \mathbf{cx} \geq \boldsymbol{z}\}$ are known as closed-half spaces.

☛ **Remarks**

- For optimum value of $z$, the hyperplane $\mathbf{cx} = \boldsymbol{z}$ is called optimal hyperplane.
- The vector $\boldsymbol{c}$ is known as vector normal to the hyperplane
- The value $\pm\dfrac{c}{|\boldsymbol{c}|}$ are called unit normals.
- The hyperplanes are always closed sets.

$$= \lambda[2u_1 + 3u_2] + (1-\lambda)[2v_1 + 3v_2]$$
$$= \lambda\cdot 7 + (1-\lambda)\cdot 7 \qquad \text{(By (1))}$$
$$= 7$$
$$\Rightarrow \qquad \boldsymbol{w} \in S$$

Hence, $S$ is a convex set.

**EXAMPLE 2.** *Show that the set* $S = \{(x_1, x_2, x_3) : 2x_1 - x_2 + x_3 \le 4\} \subset R^3$ *is convex.*

[MEERUT–2005, 12, 15]

**SOLUTION.** Let $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ be any two points of the given set $S$. Then by definition of $S$, we can write

$$2x_1 - x_2 + x_3 \le 4 \text{ and } 2y_1 - y_2 + y_3 \le 4 \qquad \ldots(2)$$

Let $\boldsymbol{w} = (w_1, w_2, w_3)$ be a point such that

$$\boldsymbol{w} = \lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y},\ 0 \le \lambda \le 1$$

which implies that

$$(w_1, w_2, w_3) = \lambda(x_1, x_2, x_3) + (1-\lambda)(y_1, y_2, y_3)$$
$$= (\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2, \lambda x_3 + (1-\lambda)y_3)$$
$$\Rightarrow \qquad w_1 = \lambda x_1 + (1-\lambda)y_1$$
$$w_2 = \lambda x_2 + (1-\lambda)y_2$$
$$\text{and} \qquad w_3 = \lambda x_3 + (1-\lambda)y_3$$

Consider $2w_1 - w_2 + w_3$

$$= \lambda(2x_1 - x_2 + x_3) + (1-\lambda)(2y_1 - y_2 + y_3)$$
$$\le 4\lambda + 4(1-\lambda) \qquad \text{(By (1))}$$
$$\le 4$$

Thus, $\qquad \boldsymbol{w} = (w_1, w_2, w_3) \in S$

Hence, $S$ is convex.

**EXAMPLE 3.** *Examine the convexity of the set*

$$S = \{(x_1, x_2) \in R^2 : 4x_1 + 3x_2 \le 6, x_1 + x_2 \ge 1\}$$

[MEERUT–1997, 2009; DELHI–2009; ASSAM–2011; PATNA–2013]

**SOLUTION.** We have $S = \{(x_1, x_2) \in R^2 : 4x_1 + 3x_2 \le 6, x_1 + x_2 \ge 1\}$

Let $\boldsymbol{u} = (x_1, x_2) \in S$ and $\boldsymbol{v} = (y_1, y_2) \in S$. Then,

$$4x_1 + 3x_2 \le 6, x_1 + x_2 \ge 1 \text{ and } 4y_1 + 3y_2 \le 6, y_1 + y_2 \ge 1 \qquad \ldots(1)$$

Let $\boldsymbol{w} = (w_1, w_2)$ be a point on the line segment joining the points $u$ and $v$, then

$$\boldsymbol{w} = \lambda\boldsymbol{u} + (1-\lambda)\boldsymbol{v}$$
$$\Rightarrow \qquad (w_1, w_2) = (\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2)$$
$$\Rightarrow \qquad w_1 = \lambda x_1 + (1-\lambda)y_1 \text{ and } w_2 = \lambda x_2 + (1-\lambda)y_2$$

Consider, $4w_1 + 3w_2 = \lambda(4x_1 + 3x_2) + (1-\lambda)(4y_1 + 3y_2)$

$$\le \lambda\cdot 6 + (1-\lambda)\cdot 6 \qquad \text{(By (1))}$$
$$\Rightarrow \qquad 4w_1 + 3w_2 \le 6 \qquad \ldots(2)$$

Also, $\qquad w_1 + w_2 = \lambda(x_1 + x_2) + (1-\lambda)(y_1 + y_2)$

$$\ge \lambda\cdot 1 + (1-\lambda)\cdot 1 \qquad \text{(Again by (1))}$$
$$\Rightarrow \qquad w_1 + w_2 \ge 1 \qquad \ldots(3)$$

Hence, from (2) and (3) we conclude that $S$ is a convex set.