Thus, the absolute error of a sum of approximate numbers is equal to the algebraic sum of their absolute error.

Now relative error $E_R = \dfrac{E_A}{X}$

$$= \frac{E_A}{X} = \frac{\delta x_1}{X} + \frac{\delta x_2}{X} + \ldots + \frac{\delta x_n}{X}$$

Maximum relative error $= \left|\dfrac{E_A}{X}\right| \le \left|\dfrac{\delta x_1}{X}\right| + \left|\dfrac{\delta x_2}{X}\right| + \ldots + \left|\dfrac{\delta x_n}{X}\right|$

ii. Error formula for subtraction:

Let $X = x_1 - x_2$

then $\delta X = \delta x_1 - \delta x_2$

$\therefore$ $E_A = \delta x_1 - \delta x_2$

$$E_R = \frac{E_A}{X} = \frac{\delta X}{X} = \frac{\delta x_1}{X} - \frac{\delta x_2}{X}$$

Though the errors δx_1 and δx_2 may be either positive or negative, however, the sum of the absolute errors is taken in order to get the maximum error. Then, the result is that absolute error of the difference of two approximate numbers may equal to the sum of their absolute errors.

Maximum absolute error $= |\delta x| \le |\delta x_1| + |\delta x_2|$

Maximum relative error $= \left|\dfrac{\delta x}{X}\right| \le \left|\dfrac{\delta x_1}{X}\right| + \left|\dfrac{\delta x_2}{X}\right|$

iii. Error formula for multiplication:

Let $X = x_1 x_2 \ldots x_n$

Since, we know that when X is a function of $x_1, x_2, \ldots, x_n$, then

$$\delta X = \frac{\partial X}{\partial x_1}\delta x_1 + \frac{\partial X}{\partial x_2}\delta x_2 + \ldots + \frac{\partial X}{\partial x_n}\delta x_n$$

$$\frac{\delta x}{X} = \frac{1}{X}\frac{\partial X}{\partial x_1}\delta x_1 + \frac{1}{X}\frac{\partial X}{\partial x_2}\delta x_2 + \ldots + \frac{1}{X}\frac{\partial X}{\partial x_n}\delta x_n$$

Now $\dfrac{\partial X}{\partial x_1} = \dfrac{\partial}{\partial x_1}(x_1 x_2 \ldots x_n) = x_2 x_3 \ldots x_n$

So that $\dfrac{1}{X}\dfrac{\partial X}{\partial x_1} = \dfrac{1}{X}\dfrac{\partial}{\partial x_1} = \dfrac{x_2 x_3 \ldots x_n}{x_1 x_2 \ldots x_n} = \dfrac{1}{x_1}$

Similarly $\dfrac{1}{X}\dfrac{\partial X}{\partial x_2} = \dfrac{1}{x_2}, \dfrac{1}{X}\dfrac{\partial X}{\partial x_3} = \dfrac{1}{x_3}$

$$\frac{1}{X}\frac{\partial X}{\partial x_n} = \frac{1}{x_n}$$

$$\frac{\partial X}{X} = \frac{\delta x_1}{x_1} + \frac{\delta x_2}{x_2} + \ldots + \frac{\delta x_n}{x_n}$$

Since the element in second row, second column is zero, interchange second and third row to get pivot element 1, i.e.

$$R(2,3) \sim \begin{pmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 1 & -3 & 0 & : & 3 \\ 0 & 0 & 2 & -3 & : & -8 \\ 0 & 1 & 0 & -2 & : & -4 \end{pmatrix}$$

$$R_4 \to R_4 - R_2 \sim \begin{pmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 1 & -3 & 0 & : & 3 \\ 0 & 0 & 2 & -3 & : & -8 \\ 0 & 0 & 3 & -2 & : & -7 \end{pmatrix}$$

Now the pivot is 2, therefore

$$R_4 \to R_4 - \frac{3}{2}R_3 \sim \begin{pmatrix} 1 & 1 & 1 & 1 & : & 2 \\ 0 & 1 & -3 & 0 & : & 3 \\ 0 & 0 & 2 & -3 & : & -8 \\ 0 & 0 & 0 & \frac{5}{2} & : & 5 \end{pmatrix}$$

$$x_1 + x_2 + x_3 + x_4 = 2$$
$$x_2 - 3x_3 = 3$$
$$2x_3 - 3x_4 = -8$$
$$\frac{5}{2}x_4 = 5$$
$$x_4 = 2$$
$$x_3 = \frac{1}{2}(-8 + 3x_4) = \frac{1}{2}(-8 + 6) = -1$$

Now by back substitution

$$x_2 = 3 + 3x_3 = 3 - 3 = 0$$
$$x_1 = 2 - x_2 - x_3 - x_4 = 2 - 0 - (-1) - 2 = 1$$
$$x_1 = 1,\ x_2 = 0,\ x_3 = -1,\ x_4 = 2$$

PROBLEMS 2.1

Solve the system of equations by Gauss elimination method.

i. $x + 2y + z = 3$
$2x + 3y + 3z = 10$
$3x - y + 2x = 13$ [Ans. $x = 2, y = -1, z = 3$]

ii. $2x + 3y - z = 5$
$4x + 4y - 3z = 3$
$2x - 3y + 2z = 2$ [Ans. $x = 1, y = 2, z = 3$]

iii. $5x_1 + x_2 + x_3 + x_4 = 4$
$x_1 + 7x_2 + x_3 + x_4 = 12$
$x_1 + x_2 + 6x_3 + x_4 = -5$
$x_1 + x_2 + x_3 + 4x_4 = 6$ [Ans. $x_1 = 1, x_2 = 2, x_3 = -1, x_4 = -2$]

This matrix is equivalent to three equations, which are equivalent to three system of equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{2.5}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.6}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.7}$$

The systems (2.5), (2.6) and (2.7) of Eqs (2.5)–(2.7) can be solve by Gauss-elimination procedure. The solution set of each system Eqs (2.5), (2.6) and (2.7) will be the corresponding column of the inverse matrix X.

Note: Since the coefficient matrix is same in all the Eqs (2.5), (2.6) and (2.7), all can be simultaneously solved by forming a definite system

$$[A/I] = \left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

Example 2.9: By Gauss elimination, find the inverse of $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & -1 & -4 \end{pmatrix}$.

Solution: The augmented system (A/I) is

$$(A/I) \sim \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right)$$

Since the element $a_{11} = 0$, we will interchange the first and second row, the reduced system is

$$(A/I) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right)$$

we get

$$R_3 \to R_3 + (-3)R_1 \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -7 & -4 & 0 & -3 & 1 \end{array}\right)$$

$$R_3 \to R_3 + 7R_2 \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 7 & -3 & 1 \end{array}\right)$$

Solving these equations, we get

$$u_{11} = 1, u_{12} = 3, u_{13} = 1$$
$$l_{21} = 1, u_{22} = 1, u_{23} = 1$$
$$l_{31} = 1, l_{32} = -1, u_{33} = -3$$

Since $LUX = B$, $LY = B$, where $UX = Y$

From $LY = B$, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$$

$\Rightarrow$
$$y_1 = 3$$
$$y_1 + y_2 = 3$$
$$y_1 - y_2 + y_3 = 6$$

$\Rightarrow y_1 = 3; y_2 = 0; y_3 = 3$

and $UX = Y$ gives

$$\begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$$

$$x + 3y + z = 3$$
$$y + z = 0$$
$$-3z = 3$$

By back substitution $z = -1$, $y = 1$, $x = 1$

$\therefore$ The required solution is $x = 1$, $y = 1$, $z = -1$.

PROBLEMS 2.5

Solve the following system of equations by method of factorisation, decomposition or triangularisation.

1. $2x + y + 3z = 13$
   $x + 5y + z = 14$
   $3x + y + 4z = 17$ [**Ans.** $x = 1; y = 2; z = 3$]
2. $x + y + 4z = 16$
   $2x - y + 3z = 16$
   $3x + y - z = -3$ [**Ans.** $x = 1; y = 2; z = 4$]
3. $3x + y + 2z = 3$
   $2x - 3y - z = -3$
   $x + 2y + z = 4$ [**Ans.** $x = 1; y = 2; z = -1$]
4. $5x - 2y + z = 4$
   $7x + y - 5z = 8$
   $3x + 7y + 4z = 4$ $\left[\text{Ans. } x = \frac{366}{327}; y = \frac{284}{327}; z = \frac{46}{327}\right]$