1.6 PROBABILITY OF AN EVENT

The probability of an event has been defined in several ways. Two of the most popular definitions are the *relative frequency* definition, and the *classical* definition.

1.6.1 The Relative Frequency Definition

Suppose that a random experiment is repeated n times. If an event A occurs n_A times, then the probability of A, denoted by $P(A)$, is defined as

$$P(A) = \lim_{n \to \infty} \left(\frac{n_A}{n} \right) \quad \text{...(1.6)}$$

where $\left(\frac{n_A}{n} \right)$ represents the fraction of occurrence of A in n trials.

1.6.2 The Classical Definition

The relative frequency definition given above has empirical flavor. In the classical approach, the probability of an event A is found without experimentation. This is done by counting the total number N of the possible outcomes of the experiment. If N_A of those outcomes are favorable to the occurrence of the event A, then

$$P(A) = \left(\frac{N_A}{N} \right) \quad \text{...(1.7)}$$

where, it is assumed that all outcomes are *equally likely.*

Axiomatic Definition of Probability

Whatever may be the definition of probability, we require the probability measure (to the various events on the sample space) to obey the following postulates or axioms:

$$\left.\begin{array}{l} \text{i. } 0 \le P(A) \le 1 \\ \text{ii. } P(S) = 1 \\ \text{iii. If } A_1, A_2, \ldots, A_n \text{ are mutually exclusive events, then} \\ \quad P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) \end{array}\right\} \quad \text{...(1.8)}$$

1.6.3 Theorems of Probability

Theorem 1: Addition rule of probability

If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

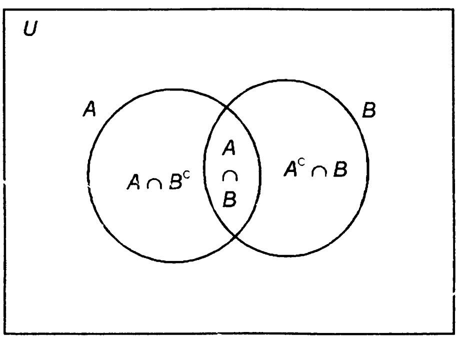

Fig. 1.3: Venn diagram of $A \cup B$

The moments m_{11}, m_{02}, m_{20} are called second order moments.
If $n = 1$ and $m = 1$,

$$m_{11} = E(XY) = \int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\,dy \qquad ...(5.6)$$

Equation (5.6) is called as the correlation of X and Y, which is denoted by R_{XY}.
If correlation can be written in the form

$$R_{XY} = E[X]E[Y] \qquad ...(5.7)$$

then X and Y are said to be uncorrelated.
If $R_{XY} = 0$, then X and Y are said to be orthogonal.
For N random variables $X_1, X_2, ..., X_N$, the $n_1 + n_2 + \cdots + n_N$ order joint moments are defined as

$$E\left[X_1^{n_1}, X_2^{n_2}, ..., X_N^{n_N}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \left(X_1^{n_1}, X_2^{n_2}, ..., X_N^{n_N}\right) f_{X_1,X_2,...,X_N}(x_1, x_2, ..., x_N)\,dx_1, dx_2, ..., dx_N \qquad ...(5.8)$$

5.2.2. Joint Moments About the Mean

The joint central moments of random variables X and Y are denoted by μ_{nk}. It is defined as

$$\mu_{nk} = E\left[(X-\overline{X})^n (Y-\overline{Y})^k\right]$$

$$\mu_{nk} = \begin{cases} \sum_j \sum_i (x_i - \overline{X})^n \left(y_j - \overline{Y}\right)^k P\left(x_i, y_j\right) & X \text{ and } Y \text{ are discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\overline{X})^n \left(y-\overline{Y}\right)^k f_{XY}(x,y)\,dx\,dy & X \text{ and } Y \text{ are continuous} \end{cases} \qquad ...(5.9)$$

Special Cases

For $n = 2$ and $k = 0$,

$$\mu_{20} = E\left[(X-\overline{X})^2\right] = \int_{-\infty}^{\infty} (x-\overline{X})^2 f_X(x)\,dx = \sigma_X^2 \qquad ...(5.10)$$

This is the variance of random variable X.
For $n = 0$ and $k = 2$,

$$\mu_{02} = E\left[(Y-\overline{Y})^2\right] = \int_{-\infty}^{\infty} \left(y-\overline{Y}\right)^2 f_Y(y)\,dy = \sigma_Y^2 \qquad ...(5.11)$$

This is the variance of random variable Y.
For $n = 1$ and $k = 1$,

$$\mu_{11} = E\left[(X-\overline{X})(Y-\overline{Y})\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\overline{X})\left(y-\overline{Y}\right) f_{XY}(x,y)\,dx\,dy \qquad ...(5.12)$$

$$= E\left[(X\cos\theta + Y\sin\theta - \overline{X}\cos\theta - \overline{Y}\sin\theta)(-X\sin\theta + Y\cos\theta + \overline{X}\sin\theta - \overline{Y}\cos\theta)\right]$$

$$= (\sigma_Y^2 - \sigma_X^2)\cos\theta\sin\theta + C_{XY}[\cos^2\theta - \sin^2\theta]$$

$$C_{Y_1Y_2} = (\sigma_Y^2 - \sigma_X^2)\frac{1}{2}\sin 2\theta + C_{XY}\cos 2\theta$$

If we require Y_1 and Y_2 being uncorrelated, we must have $C_{Y_1Y_2} = 0$

$$0 = (\sigma_Y^2 - \sigma_X^2)\frac{1}{2}\sin 2\theta + C_{XY}\cos 2\theta$$

But $$C_{XY} = \rho\sigma_X\sigma_Y$$

$$(\sigma_Y^2 - \sigma_X^2)\frac{1}{2}\sin 2\theta = -\rho\sigma_X\sigma_Y\cos 2\theta$$

$$\frac{1}{2}\tan 2\theta = \frac{-\rho\sigma_X\sigma_Y}{(\sigma_Y^2 - \sigma_X^2)}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_X\sigma_Y}{(\sigma_X^2 - \sigma_Y^2)}\right)$$

Some important properties of Gaussian random variables are:

1. By using first and second order moments only, we can define Gaussian random variables completely.
2. The linear combination of two independent random variables having a normal distribution also have a normal distribution.
3. Uncorrelated random variables are also statistically independent.
4. Any K-dimensional marginal density function obtained from N-dimensional density function by integrating N-K random variables will be Gaussian.

Example 5.3.1: Statistically independent random variables X and Y have moments $m_{10} = 2$, $m_{20} = 14$, $m_{02} = 12$, $m_{11} = -6$. Find the moment m_{22}.

Solution: $m_{10} = E[X] = 2$, $m_{20} = E[X^2] = 14$, $m_{02} = E[Y^2] = 12$ and $m_{11} = E[XY] = -6$

For statistically independent random variables, $m_{11} = E[XY] = E[X]E[Y] = -6$

$$E[Y] = -\frac{6}{2} = -3$$

$$m_{22} = E\left[(x-\overline{X})^2(y-\overline{Y})^2\right] = E\left[x^2 - 2x\overline{X} + \overline{X}^2\right]E\left[y^2 - 2y\overline{Y} + \overline{Y}^2\right]$$

$$= \left(E[x^2] - 2\overline{X}E[x] + \overline{X}^2\right)\left(E[y^2] - 2\overline{Y}E[y] + \overline{Y}^2\right) = (14-4)(12-9) = 30$$

Example 5.3.2: The density function of two random variables X and Y is

$$f_{XY}(x,y) = \begin{cases} \dfrac{1}{24} & 0 < x < 6, 0 < y < 4 \\ 0 & \text{otherwise} \end{cases}$$

Find the expectation of the function $g(x, y) = (xy)^2$.

Solution: $E[g(x,y)] = \int\limits_0^4\int\limits_0^6 \frac{1}{24}(xy)^2\,dxdy = \frac{1}{24}\frac{x^3}{3}\bigg|_0^6 \frac{y^3}{3}\bigg|_0^4 = 64$

$$E[X] = \frac{1+2+3+4+5+6+7+8+9}{9} = 5$$

$$E[Y] = \frac{8+9+10+11+12+13+14+15+16}{9} = 12$$

$$E[XY] = \frac{1*8+2*9+3*10+4*11+5*12+6*13+7*14+8*15+9*16}{9} = 66.667$$

$$E[X^2] = \frac{1+4+9+16+25+36+49+64+81}{9} = 31.667$$

$$E[Y^2] = \frac{64+81+100+121+144+169+196+225+256}{9} = 150.667$$

$$\text{Var}(X) = E[X^2]-(E[X])^2 = 31.667-25 = 6.667$$

$$\text{Var}(Y) = E[Y^2]-(E[Y])^2 = 150.667-144 = 6.667$$

$$C_{XY} = E[XY]-E[X]E[Y] = 66.667-60 = 6.667$$

$$\rho = \frac{C_{XY}}{\sigma_X\sigma_Y} = \frac{6.667}{\sqrt{(6.667)(6.667)}} = 1.$$

Example 5.13: Let X and Y be two random variables having $\overline{X} = 1, \overline{Y} = 2, \sigma_X^2 = 6$, $\sigma_Y^2 = 9$ and $\rho = \frac{2}{3}$. Find: (a) covariance of X and Y (b) correlation of X and Y (c) moments m_{02} and m_{20}.

Solution: Correlation of X and Y is

$$C_{XY} = \rho\sigma_X\sigma_Y = \frac{2}{3}*\sqrt{6}*3 = 2\sqrt{6}$$

$$R_{XY} = C_{XY}+E[X]E[Y] = 2\sqrt{6}+2 = 2(1+\sqrt{6})$$

$$m_{20} = \sigma_X^2+(E[X])^2 = 6+1 = 7$$

$$m_{02} = \sigma_Y^2+(E[Y])^2 = 9+4 = 13.$$

Example 5.14: Two discrete random variables X and Y have the joint density function $f_{XY}(x, y) = 0.4\delta(x+a)\,\delta(y-2) + 0.3\delta(x-a)\,\delta(y-2) + 0.18\delta(x-a)\,\delta(y-a) + 0.2\delta(x-1)\,\delta(y-1)$. Determine the value of a, if any, that minimizes the correlation between X and Y and also find the minimum correlation.

Solution: $R_{XY} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x,y)dx\,dy$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\left[0.4\delta(x+a)\delta(y-2)+0.3\delta(x-a)\delta(y-2)+0.18\delta(x-a)\delta(y-a)+0.2\delta(x-1)\delta(y-1)\right]dxdy$$

$$\because \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(g_1,g_2)\delta(g_1-x)\delta(g_2-2)dg_1\,dg_2 = Q(x,y)$$

$$R_{XY} = 0.4(-a)(2)+0.3(a)(2)+0.18(a)(a)+0.2(1)(1) = 0.1a^2-0.2a+0.2$$