second order determinant, which is called the minor of the element a_{ij}. It is denoted by M_{ij} . Therefore, in a determinant of order 3, we may get 9 minors corresponding to the 9 elements of the determinant.

For example, in determinant (1)

$$\text{Minor of } a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$$

and

$$\text{Minor of } a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = M_{32}$$

If we expand the determinant along the first row, then

$$\Delta = (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} + (-1)^{1+3} a_{13} M_{13}$$
$$= a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

Similarly, along second column, we can write

$$\Delta = -a_{12} M_{12} + a_{22} M_{22} - a_{32} M_{32}$$

1.11.2 Cofactor

If we multiply the minor M_{ij} by $(-1)^{i+j}$. Then resulting value is called cofactor of the element a_{ij}. If A_{ij} is the cofactor of a_{ij}, then we write

$$\text{Cofactor of } a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$$

$$\text{Cofactor of } a_{21} = A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{Cofactor of } a_{32} = A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Hence, cofactor of a_{ij} = $(-1)^{i+j}$ determinant obtained by leaving row and column passing through that element. Therefore, we can write

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$
$$\Delta = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

and

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$$
$$a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$$

1.12 SINGULAR AND NON-SINGULAR MATRIX

Definition. *A matrix whose determinant value is zero, is said to be singular matrix.*

If the matrix is not singular, then it is said to be non-singular.

For example : If $A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$, then its determinant value.

$$|A| = \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 2 \times 9 - 3 \times 6 = 18 - 18 = 0$$

Thus the matrix A is singular.

1.13 TRANSPOSE OF A MATRIX

Consider a matrix $A = [a_{ij}]_{m \times n}$. Then a matrix which is obtained by interchanging the rows and columns of A is called the transpose of A. It is denoted by A' or A^T.

Applying $R_2 \to R_2 - 3R_1$, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} A$$

Again applying $R_1 \to R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} A$$

$\Rightarrow$ $$I_2 = BA$$

$\Rightarrow$ $$A^{-1} = B = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}.$$

(ii) We write

$$A = I_2 A$$

or $$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

Applying $R_2 \to -\frac{1}{5} R_2$, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2/5 & -1/5 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix} A$$

$\Rightarrow$ $$I_2 = BA$$

$\Rightarrow$ $$A^{-1} = B = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}$$

EXAMPLE 2. *Find the inverse of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

by using elementary row-transformation.

SOLUTION. We write $$A = I_3 A$$

or $$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

☛ **REMARK**

- A function $f(\boldsymbol{x})$ is said to be strictly concave if $-f(\boldsymbol{x})$ is strictly convex.

RELATED THEOREMS

THEOREM 1. *The hyperplane is a convex set.* [MEERUT–2007]

PROOF. Let $X = [\boldsymbol{x} : \mathbf{c}\boldsymbol{x} = z]$ be a hyperplane and $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$ then,

$$\mathbf{c}\boldsymbol{x}_1 = z \text{ and } \mathbf{c}\boldsymbol{x}_2 = z \qquad \text{(By definition)}$$

Now, if $\boldsymbol{x}_3 = \lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2,\ 0 \le \lambda \le 1$

Then,
$$c\boldsymbol{x}_3 = \lambda c\cdot\boldsymbol{x}_1 + (1-\lambda)c\boldsymbol{x}_2$$
$$= \lambda z + (1-\lambda)z$$
$$= z$$

$$\Rightarrow \quad \boldsymbol{x}_3 = \lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in X$$

$\Rightarrow \boldsymbol{x}_3$ is also a point in X

Hence, X is a convex set.

THEOREM 2. *The closed half spaces $H_1 = \{\boldsymbol{x} : \mathbf{c}\boldsymbol{x} \ge z\}$ and $H_2 = \{\boldsymbol{x} : \mathbf{c}\boldsymbol{x} \le z\}$ are convex sets.*

PROOF. Let $\boldsymbol{x}_1 \in H_1$ and $\boldsymbol{x}_2 \in H_2$. Then by definition of H_1, we can write $\mathbf{c}\boldsymbol{x}_1 \ge z : \mathbf{c}\boldsymbol{x}_2 \ge z$

Now, if $0 \le \lambda \le 1$, then we have

$$\mathbf{c}[\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2] = \lambda\mathbf{c}\cdot\boldsymbol{x}_1 + (1-\lambda)\mathbf{c}\boldsymbol{x}_2$$
$$\ge \lambda z + (1-\lambda)z = z$$

Therefore, $\boldsymbol{x}_1, \boldsymbol{x}_2 \in H_1$ and $0 \le \lambda \le 1$ implies $\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in H_1$

Hence, H_1 is a convex set

Similarly, we may prove that H_2 is a convex set.

☛ **REMARK**

- In a similar way (as above) we may prove that the open half spaces $\{\boldsymbol{x} : \mathbf{c}\boldsymbol{x} > z\}$ and $\{\boldsymbol{x} : \mathbf{c}\boldsymbol{x} < z\}$ are convex sets.

THEOREM 3. *Intersection of two convex sets is also a convex set.* [MEERUT–2007, 08, 12,15]

PROOF. Let X_1 and X_2 be two convex sets. We have to prove that $X_1 \cap X_2$ is also convex.

If $\boldsymbol{x}_1 \in X_1 \cap X_2 \Rightarrow \boldsymbol{x}_1 \in X_1$ and $\boldsymbol{x}_1 \in X_2$

$\boldsymbol{x}_2 \in X_1 \cap X_2 \Rightarrow \boldsymbol{x}_2 \in X_1$ and $\boldsymbol{x}_2 \in X_2$

Now, by definition of convex sets

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in X_1 \Rightarrow \lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in X_1 \ ;\ 0 \le \lambda \le 1$$

$$\boldsymbol{x}_1, x_2 \in X_2 \Rightarrow \lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in X_2 \ ;\ 0 \le \lambda \le 1$$

Therefore, $\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in X_1$ and $\lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in X_2$

$\Rightarrow \lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in X_1 \cap X_2$

Hence, $X_1 \cap X_2$ is a convex set.

THEOREM 4. *Finite intersection of convex sets is also a convex set.*

PROOF. Let $X_1, X_2, \ldots, X_n$ be n convex sets.

We have to prove that $X = X_1 \cap X_2 \cap \ldots \cap X_n$ is also convex.

Let $\boldsymbol{x}_1 \in X_1 \cap X_2 \cap \ldots \cap X_n \Rightarrow \boldsymbol{x}_1 \in X_i \forall i = 1, 2, \ldots, n$

$\boldsymbol{x}_2 \in X_1 \cap X_2 \cap \ldots \cap X_n \Rightarrow \boldsymbol{x}_2 \in X_i \forall i = 1, 2, \ldots, n$

Glossary

- **Non-degenerate Basic solution:** If none of the basic variable is zero. Then, basic solution is called non-degenerate.
- **Degenerate Basic solution:** If at least one of the basic variable is zero, then a basic solution is called degenerate.
- **Feasible Basic solution:** If all the basic variables are non-negative, then a basic solution is called feasible.
- **Hypersphere:** In n-dimensional space, a hypersphere, with centre a and radius $r(>0)$ is the set of points
$$X = \{x : |x - a| = r\}$$
The equation of hypersphere in E^n (or R^n) is given by $\Sigma(x_i - a_i)^2 = r^2$.
- **Hyperplane:** It is defined as the set of points $(x_1, x_2, ..., x_n)$ satisfying $c_1x_1 + c_2x_2 + ... + c_nx_n = z$, (not all $c_i = 0$) for prescribed values of $c_1,c_2,..., c_n$ and z.
- **Convex set:** A set of points is said to be convex if for any two points in the set, the line segment joining these points is also in the set, *i.e.*, a set is said to be convex if convex combination of any two points in the set is also in the set.
- **Convex Hull:** The set of all convex combinations of sets of points from the set X of points is called convex hull, *i.e.*, the intersection of all convex sets containing X in n-dimensional space is called the convex hull of X. Hence, the convex hull of a set $X \subseteq E^n$ is the smallest convex set containing X.
- **Convex function:** A function $f(\boldsymbol{x})$ is said to be strictly convex at $\boldsymbol{x}$ if for any two other distinct points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$
$$f\{\lambda x_1 + (1 - \lambda)x_2\} < \lambda f(x_1) + (1 - \lambda) f(x_2), \quad 0 < \lambda < 1$$
- **Convex Polyhedron:** The set of all convex combinations of finite number of points is said to be the convex polyhedron generated by these points.
- **Extreme Point:** A point $\boldsymbol{x}$ in a convex set C is an extreme point of C if it does not lie on the line segment of any two points, different from $\boldsymbol{x}$ in the set, *i.e.*, it can not be expressed as a convex combinations of any two distinct points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in C.

REVIEW QUESTIONS

1. What do you mean by an extreme point of a convex set?
2. Write a short note on convex set and their applications to linear programming problem.
3. Obtain the convex hull of the boundary of a circle.
4. Prove that the convex hull of a finite number of points is a convex set.
5. Define: Hyperplane, Convex set
6. What is meant by convex polyhedron.
7. Explain the procedure of generating extreme points solutions to a linear programming problem pointing out the assumption made. if any?

MULTIPLE CHOICE QUESTIONS (CHOOSE THE MOST APPROPRIATE ONE)

1. The number of vertices of any non empty closed bounded convex set can not be:
(a) finite (b) not finite
(c) infinite (d) None of these
2. The closed half spaces in E_n or E^n is a:
(a) open convex set
(b) unbounded convex set
(c) closed convex set
(d) no convex set
3. The set of all feasible solution (if not empty) of a L.P.P. is a:
(a) non convex set (b) poly convex set
(c) convex set (d) none of these
4. The union of two convex sets may or may not be a:
(a) Non convex set (b) Convex set
(c) Poly convex set (d) None of these
5. Every extreme point of a convex set is:
(a) boundary value of the set
(b) boundary point of the set
(c) both (a) and (b)
(d) none of these
6. A hyper plane is:
(a) convex (b) feasible
(c) concave (d) none of these
7. Let S and T are two convex sets in E^n, then $S + T$, $S - T$ and $\alpha S + \beta T$, where α and β are scalars, are called:
(a) non convex (b) convex sets
(c) convex point (d) none of these