(iv) $F(t) = e^{at}$ (Meerut–1991, 2004; Andhra–1990; Purvanchal–1989, 96)

SOLUTION. We have $L\{F(t)\} = \int_0^\infty e^{-pt} e^{at}\, dt = \int_0^\infty e^{-(p-a)t}\, dt \cdot$

If $p \le a$, integral diverges. For $p > a$, the integral converges. Hence, for $p > a$,

$$L\{e^{at}\} = \int_0^\infty e^{-(p-a)t}\, dt = \left[-\frac{e^{-(p-a)t}}{p-a}\right]_0^\infty = 0 + \frac{1}{p-a}$$

$$= \frac{1}{p-a},\ \ p > a \cdot$$

(v) $F(t) = \sin at$ (Raj.–1983; Kanpur–1984, 95, 2002, 03, 04; Meerut–1991, 2005, 08; Purvanchal–1989; MS Univ.(T.N.)–2007)

SOLUTION. We have $L\{\sin at\} = \int_0^\infty e^{-pt} \sin at\, dt = \left[\frac{e^{-pt}(-p\sin at - a\cos at)}{p^2 + a^2}\right]_0^\infty$

$$\left[\because \int e^{ax} \sin bx dx = e^{ax} \frac{[a\sin bx - b\cos bx]}{a^2 + b^2}\right]$$

$$= \frac{a}{p^2 + a^2},\ \ p > a$$

Hence, $L\{\sin at\} = \dfrac{a}{p^2 + a^2}$.

(vi) $F(t) = \cos at$ (Meerut–1991; Rohilkhand–2007; Kanpur–1995; MS Univ.(T.N.)–2007)

SOLUTION. We know that

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}\ (a\cos bx + b\sin bx)}{a^2 + b^2}$$

Therefore, we have

$$L\{\cos at\} = \int_0^\infty e^{-pt} \cos at\, dt = \left[\frac{e^{-pt}(-p\cos at + a\sin at)}{a^2 + p^2}\right]_0^\infty$$

$$= \frac{p}{p^2 + a^2},\ \ p > 0.$$

(vii) $F(t) = \sinh at$ (Meerut–1980, 83, 1991; 2009; Rohilkhand–1988; Andhra–1990; MS Univ.(T.N.)–2007)

SOLUTION. Consider $L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\} = \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\}$ [Using (iv)]

$$= \frac{1}{2}.\frac{1}{p-a} - \frac{1}{2}.\frac{1}{p+a} = \frac{a}{p^2 - a^2}$$

Hence, $L\{\sinh at\} = \dfrac{a}{p^2 - a^2}$.

(viii) $F(t) = \cosh at$ (Meerut–1978, 80, 83; Rohilkhand–1988; Andhra–1990; Purvanchal–2001; Osmania–2004; Sagar–2004; MS Univ.(T.N.)–2007)

SOLUTION. Consider

$$L\{\cosh at\} = L\left[\frac{1}{2}(e^{at} + e^{-at})\right] = \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$$

$$\Rightarrow \qquad \sin^3 2t = \frac{1}{4}[3\sin 2t - \sin 6t]$$

$$\therefore \qquad L\{\sin^3 2t\} = \frac{1}{4}L\{3\sin 2t - \sin 6t\} = \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\}$$

$$= \frac{3}{4}\left[\frac{2}{p^2+2^2}\right] - \frac{1}{4}\left[\frac{6}{p^2+6^2}\right], p > 0$$

$$= \frac{3}{2}\left(\frac{1}{p^2+4}\right) - \frac{3}{2}\left[\frac{1}{p^2+36}\right] = 48/[(p^2+4)(p^2+36)]$$

Additional Solved Examples

EXAMPLE 1. *Find* $L\{\sin t \cos t\}$. (Meerut–2012)

SOLUTION. We have

$$L\{\sin t\cos t\} = L\left\{\frac{1}{2}\sin 2t\right\} = \frac{1}{2}L\{\sin 2t\} \qquad (\because \sin 2t = 2\sin t\cos t)$$

$$= \frac{1}{2}\cdot\frac{2}{p^2+2^2}, p > 0 \qquad \left[\because L\{\sin at\} = \frac{a}{p^2+a^2}\right]$$

$$= \frac{1}{p^2+4}, p > 0.$$

EXAMPLE 2. *Find* $L\{4\cos^2 t\}$.

SOLUTION. We have $L\{4\cos^2 t\} = 4L[\cos^2 t] = 2L\{1+\cos 2t\}$ $\qquad (\because \cos 2t = 2\cos^2 t - 1)$

$$= 2\left[\frac{1}{p} + \frac{p}{p^2+2^2}\right], p > 0$$

$$= \frac{2(2p^2+4)}{p(p^2+4)}, p > 0 = \frac{4p^2+8}{p(p^2+4)}, p > 0$$

EXAMPLE 3. *Find* $L\{\sin^2 at\}$.

(Meerut–1991, 94; Purvanchal–2002, 05; Kanpur–1980; Nagarjuna–2006)

SOLUTION. We have

$$L(\sin^2 at) = L\left\{\frac{1}{2}(1-\cos 2at)\right\} = \frac{1}{2}[L\{1\} - L\{\cos 2at\}]$$

$$= \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+(2a)^2}\right], p > 0 = \frac{2a^2}{p(p^2+4a^2)}, p > 0$$

EXAMPLE 4. *Find* $L\{3\cosh 5t - 4\sinh 5t\}$

SOLUTION. We have

$$L\{3\cosh 5t - 4\sinh 5t\} = 3L\{\cosh 5t\} - 4L\{\sinh 5t\}$$

$$= 3\left[\frac{p}{p^2-5^2}\right] - 4\left[\frac{5}{p^2-5^2}\right] = \frac{3p-20}{p^2-25}, p > 5$$

EXAMPLE 5. *Find* $L\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$

SOLUTION. We have

$$L\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$$

We have $L\{F(t)\} = L\{\cos t\} = \dfrac{p}{p^2+1} = f(p)$ (say)

Using second translation or shifting theorem, we have

$$L\{G(t)\} = e^{\left(-\frac{2\pi}{3}\right).p} .f(p) = e^{-2\pi p/3} \cdot \frac{p}{p^2+1}.$$

EXAMPLE 9. *Find* $L\{G(t)\}$, *where* $G(t) = \begin{cases} e^{t-a}, & t > a \\ 0, & t < a \end{cases}.$ (UPTU–2008)

SOLUTION. By second shifting theorem, we have

If $L\{F(t)\} = f(p)$ and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$. Then, $L\{G(t)\} = e^{-ap} f(p)$

Let $F(t) = e^t$

Then, $L\{F(t)\} = L\{e^t\} = \int_0^\infty e^{-pt}.e^t dt = \dfrac{1}{p-1}, p > 1 = f(p)$ (say)

Now, let $G(t) = \begin{cases} F(t-a) = e^{t-a}, & t > a \\ 0, & t < a \end{cases}$

Then, $L\{G(t)\} = e^{-ap} f(p) = \dfrac{e^{-ap}}{p-1}, \; p > 1$.

EXAMPLE 10. *Evaluate* $L\{e^{-t}\cos^2 t\}$ (Bangalore–1985)

SOLUTION. Here, we have

$$L\left\{e^{-t}\cos^2 t\right\} = L\left\{\frac{1}{2}e^{-t}(1+\cos 2t)\right\} = \frac{1}{2}L\left\{e^{-t}\right\} - \frac{1}{2}L\left\{e^{-t}\cos 2t\right\}$$

$$= \frac{1}{2}\left[\frac{1}{p+1} + \frac{(p+1)}{(p+1)^2+2^2}\right] = \frac{1}{2}\left[\frac{(p^2+2p+5)+(p+1)^2}{(p+1)(p^2+2p+5)}\right]$$

$$= \frac{p^2+2p+3}{(p+1)(p^2+2p+5)}$$

EXAMPLE 11. *Evaluate* $L\{t^5 e^{3t}\}$ (Kanpur–2004)

SOLUTION. Here, we have

$$L\{t^5\} = \frac{\Gamma(n+1)}{p^{n+1}} \text{ if } n = 5.$$

$$\Rightarrow \quad L\{t^5\} = \frac{\Gamma(6)}{p^6} = \frac{5!}{p^6}$$

$$\Rightarrow \quad L\{t^5 e^{3t}\} = \frac{120}{(p-3)^6} \text{ as } L\{e^{at}F(t)\} = f(p-a)$$

EXAMPLE 12. *Prove that* $L\{e^{2t}(\cos 4t + 3\sin 4t)\} = \dfrac{p+10}{p^2-4p+20}$.

SOLUTION. Here, we have

$$L\{e^{2t}(\cos 4t + 3\sin 4t)\}$$

The series (1) with coefficients a_0, a_n and b_n given by (2), (3) and (4) respectively is called the Fourier series of $f(x)$ and the coefficients a_0, a_n and b_n are called the Fourier coefficients corresponding to $f(x)$.

(i) When $c = 0$, the interval becomes $0 < x < 2\pi$ and formula for a_0, a_n, b_n is obtained by putting $c = 0$.

(ii) When $c = -\pi$, then interval becomes $-\pi < x < \pi$. In this interval, the formula for a_0, a_n and b_n becomes as under :

(a) When $f(x)$ is an odd function, then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0 \cdot \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0$$

[By property of definite integral]

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x dx$$

Hence, if function $f(x)$ is odd, its Fourier expansion contains only sine series,

i.e., $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \text{where}\, b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx.$$

(b) When $f(x)$ is even function, then formula for a_0, a_n and b_n are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} f(x)dx\,,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = 0$$ [$\because$ $f(x)$ sin nx is odd.]

Hence, if a periodic function $f(x)$ is even, its Fourier expansion contains only cosine terms, *i.e.*, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\int_0^{\pi} f(x)dx$, where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \text{ and } a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos nx\, dx$$

9.3 SOME IMPORTANT RESULTS

The following results are useful in the Fourier series :

(i) $\sin n\pi = 0, \cos n\pi = (-1)^n, \cos\left(n + \frac{1}{2}\right)\pi = 0,$ where $n \in Z$.

(ii) $\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots,$ where $u' = \frac{du}{dx}, u'' = \frac{d^2u}{dx^2}, \ldots$

$$v_1 = \int v dx, v_2 = \int v_1 dx, \ldots$$