2.3.3 Column Matrix

A matrix having m rows and only one column is called a *column matrix* of order $m \times 1$.

For example : $A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}_{m\times 1}$

2.3.4 Horizontal Matrix

A matrix having more columns than the number of its rows, is called *Horizontal matrix*.

For example: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$

2.3.5 Vertical Matrix

A matrix having more number of rows than its columns, is called *vertical matrix*.

For exmaple: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2}$

☛ **Remark**

- Row matrix is also a horizontal matrix and column matrix is also a vertical matrix.

2.3.6 Square Matrix

A matrix having a number of rows equal to number of columns, is called *square matrix*.

For example : $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$

Here, the matrix A has 3 rows and 3 columns, so it is a square matrix. Also the elements a_{11}, a_{22}, a_{33} are placed in the diagonal, so these elements are known as *diagonal elements*.

2.3.7 Diagonal Matrix

A matrix of order $n \times n$ is called a *diagonal matrix* if it contains all its off diagonal elements equal to zero.

Suppose $A = [a_{ij}]_{n\times n}$ and if $a_{ij} = 0$ for all $i \neq j$, then A is a diagonal matrix. Diagonal matrix of order $n\times n$ is usually written as Diag $[a_{11} \quad a_{22} \quad a_{33} \quad \ldots \quad a_{nn}]$

For example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3} = \text{Diag } [1 \;\; 2 \;\; 3]$

2.3.8 Scalar Matrix

A diagonal matrix whose diagonal elements are all equal but not equal to 1 is called a *scalar matrix*.

For example: $A = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}, k \neq 1$

In other words, the rank of a matrix is the order of any highest order of a non-zero minor of the matrix.

REMARKS

- If the order of a matrix A is $m \times n$, then $\rho(A) \leq \min. \{m, n\}$
- A is a null matrix iff $\rho(A) = 0$.
- If A is any non-zero matrix, then $\rho(A) \geq 1$.
- $\rho(A) \geq r$, if there exists a non-zero r-minor of A.
- For any square matrix A of order n, $\rho(A) = n$ iff A is non-singular.
- For any square matrix A of order n, $\rho(A) < n$ iff A is singular.
- $\rho(A) \leq r$ if every s-minor of A is zero, where $s > r$.

> Every $(r+1)$- rowed minor of A can be expressed as a linear combination of its r-rowed minors, therefore if every r-minor of A is zero, then its every $(r+1)$-minor is also zero.

2.18 ECHELON FORM OF A MATRIX

A matrix A is said to be in Echelon form if :

(*i*) every row of A has all its entries 0 which occurs below every row having a non-zero entry. and

(*ii*) the number of zeros before the first non-zero entry in a row is less than the number of such zeros in the next row.

REMARK

- The rank of a matrix is equal to the number of non-zero rows in Echelon form of that matrix.

For example: Consider a matrix $A = \begin{bmatrix} 0 & 2 & 3 & 5 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clearly, A is in Echelon form which has 2 non-zero rows, hence the rank of A is 2.

THEOREM 1. ***The rank of the transpose of a matrix is equal to the rank of that matrix.***

PROOF. Let A be a marix, then A' is its transpose and let $\rho(A) = r$, then there exists an r-rowed minor of A which is not equal to zero and all s-rowed minors of A are zero, where $s > r$. Let $|B|$ be a r-rowed minor of A such that $|B| \neq 0$. Since A' is the transpose of A, then $|B'|$ is the r-rowed minor of A' but $|B'| = |B| \neq 0$, therefore $\rho(A') \geq r$. Suppose there is an s-minor $|C|$ of A' such that $|C| \neq 0$, where $s > r$, then $|C'|$ will be an s-minor of A such that $|C'| = |C| \neq 0$, therefore $\rho(A) > r$ which is a contradiction, hence $\rho(A') = r$.

Solved Examples

EXAMPLE 1. ***Find the rank of the following matrices :***

(i) $\begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 1 & 5 & 2 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

REMARK

- A function $f(\boldsymbol{x})$ is said to be strictly concave if $-f(\boldsymbol{x})$ is strictly convex.

THEOREM 1. ***The hyperplane is a convex set.*** [MEERUT–2007]

PROOF. Let $X = [\boldsymbol{x} : \boldsymbol{cx} = z]$ be a hyperplane and $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$ then,

$$\boldsymbol{cx}_1 = z \text{ and } \boldsymbol{cx}_2 = z \qquad \text{(By definition)}$$

Now, if $\quad \boldsymbol{x}_3 = \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2,\ 0 \le \lambda \le 1$

Then,
$$\begin{aligned} \boldsymbol{cx}_3 &= \lambda \boldsymbol{c}\cdot\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{cx}_2 \\ &= \lambda z + (1-\lambda)z \\ &= z \end{aligned}$$

$\Rightarrow \quad \boldsymbol{x}_3 = \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in X$

$\Rightarrow \boldsymbol{x}_3$ is also a point in X

Hence, X is a convex set.

THEOREM 2. ***The closed half spaces $H_1 = \{x : cx \ge z\}$ and $H_2 = \{x : cx \le z\}$ are convex sets.***

PROOF. Let $\boldsymbol{x}_1 \in H_1$ and $\boldsymbol{x}_2 \in H_2$. Then by definition of H_1, we can write $\boldsymbol{cx}_1 \ge z : \boldsymbol{cx}_2 \ge z$

Now, if $0 \le \lambda \le 1$, then we have

$$\begin{aligned} \mathbf{c}[\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2] &= \lambda \mathbf{c}\cdot\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{cx}_2 \\ &\ge \lambda z + (1-\lambda)z = z \end{aligned}$$

Therefore, $\boldsymbol{x}_1, \boldsymbol{x}_2 \in H_1$ and $0 \le \lambda \le 1$ implies $\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in H_1$

Hence, H_1 is a convex set

Similarly, we may prove that H_2 is a convex set.

REMARK

- In a similar way (as above) we may prove that the open half spaces $\{\boldsymbol{x} : \boldsymbol{cx} > z\}$ and $\{\boldsymbol{x} : \boldsymbol{cx} < z\}$ are convex sets.

THEOREM 3. ***Intersection of two convex sets is also a convex set.***

[MEERUT–2007, 08, 12,15, 17, 18]

PROOF. Let X_1 and X_2 be two convex sets. We have to prove that $X_1 \cap X_2$ is also convex.

If $\boldsymbol{x}_1 \in X_1 \cap X_2 \Rightarrow \boldsymbol{x}_1 \in X_1$ and $\boldsymbol{x}_1 \in X_2$

$\boldsymbol{x}_2 \in X_1 \cap X_2 \Rightarrow \boldsymbol{x}_2 \in X_1$ and $\boldsymbol{x}_2 \in X_2$

Now, by definition of convex sets

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in X_1 \Rightarrow \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in X_1 \ ;\ 0 \le \lambda \le 1$$

$$\boldsymbol{x}_1, x_2 \in X_2 \Rightarrow \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in X_2 \ ;\ 0 \le \lambda \le 1$$

Therefore, $\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in X_1$ and $\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in X_2$

$\Rightarrow \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 \in X_1 \cap X_2$

Hence, $X_1 \cap X_2$ is a convex set.

THEOREM 4. ***Finite intersection of convex sets is also a convex set.*** [MEERUT–2016]

PROOF. Let $X_1, X_2, \ldots, X_n$ be n convex sets.

We have to prove that $X_1 \cap X_2 \cap \ldots \cap X_n$ is also convex.

Let $\boldsymbol{x}_1 \in X_1 \cap X_2 \cap \ldots \cap X_n \Rightarrow \boldsymbol{x}_1 \in X_i \forall i = 1, 2, \ldots, n$

$\boldsymbol{x}_2 \in X_1 \cap X_2 \cap \ldots \cap X_n \Rightarrow \boldsymbol{x}_2 \in X_i \forall i = 1, 2, \ldots, n$

Since each X_i is convex set for $i = 1, 2, \ldots, n$

(the convex set of all feasible solutions of $A\boldsymbol{x} = \boldsymbol{b}$) such that $\boldsymbol{x} = \lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2$, $0 < \lambda < 1$ ...(3)

But we can write,

$$\boldsymbol{x}_1 = [\boldsymbol{u}_1, \boldsymbol{v}_1] \text{ and } \boldsymbol{x}_2 = [\boldsymbol{u}_2, \boldsymbol{v}_2] \qquad ...(4)$$

where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are vectors of m components of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ respectively and $\boldsymbol{v}_1, \boldsymbol{v}_2$ are $(n\text{-}m)$ components vectors.

Now using (1) and (4) in (3) we get

$$[X_B, \mathbf{0}] = \lambda[\boldsymbol{u}_1, \boldsymbol{v}_1] + (1-\lambda)[\boldsymbol{u}_2, \boldsymbol{v}_2], 0 < \lambda < 1$$

$$= [\lambda\boldsymbol{u}_1 + (1-\lambda)\boldsymbol{u}_2, \lambda\boldsymbol{v}_1 + (1-\lambda)\boldsymbol{v}_2]$$

$$\Rightarrow \quad \mathbf{0} = \lambda\boldsymbol{v}_1 + (1-\lambda)\boldsymbol{v}_2,\ 0 < \lambda < 1 \qquad ...(5)$$

Clearly, $1 > \lambda > 0$, $1 - \lambda > 0$ and the components of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are greater than equal to 0. So, (5) is satisfied only when $\boldsymbol{v}_1 = 0$ and $\boldsymbol{v}_2 = 0$

So, $$\boldsymbol{x}_1 = [\boldsymbol{u}_1, 0], \boldsymbol{x}_2 = [\boldsymbol{u}_2, 0]$$

Also, $\boldsymbol{x}_1 \in X, \boldsymbol{x}_2 \in X$, then from (2) we have

$$A\,\boldsymbol{x}_1 = B\,\boldsymbol{u}_1 = \boldsymbol{b}$$

and $$A\,\boldsymbol{x}_2 = B\,\boldsymbol{u}_2 = \boldsymbol{b}$$

$$\Rightarrow \quad X_B = \boldsymbol{u}_1 = \boldsymbol{u}_2$$

So, $\boldsymbol{x} = x_1 = x_2$, which contradict the fact that $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$

$\Rightarrow \boldsymbol{x}$ cannot be expressed as a convex combinations of any two distinct points in the set of all feasible solutions.

Hence, $\boldsymbol{x}$ is an extreme point.

Conversly, let us suppose that $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ be an extreme point. We have to prove that $\boldsymbol{x}$ is a BFS.

Here, it is sufficient to prove that the vector associated with the positive elements of $\boldsymbol{x}$ are linearly independent.

Let us suppose p-components in $\boldsymbol{x}$ are non-zero and $(n\text{-}p)$ components are zero.

We can assume these components as the first p-components of $\boldsymbol{x}$.

$$\text{So, } \sum_{i=1}^{p} \alpha_i x_i = \boldsymbol{b},\ x_i > 0,\ \text{i} = 1, 2, ..., p \qquad ...(6)$$

where α_i is the column vector in A associated to the i^{th} variable in $\boldsymbol{x}$

Let if possible the column vectors $\alpha_1, \alpha_2, ..., \alpha_p$ of matrix be linearly dependent. Then by definition, there exist some scalars λ_i $(i = 1, 2, ..., p)$ with at least one of them non-zero such that $\sum_{i=1}^{p} \lambda_i \alpha_i = \mathbf{0}$ ...(7)

Now, from (6) and (7), for some arbitrary $\delta > 0$ we have

$$\sum_{i=1}^{p} x_i\alpha_i \pm \delta \sum_{i=1}^{p} \lambda_i \alpha_i = \boldsymbol{b}$$

$$\Rightarrow \quad \sum_{i=1}^{p} (x_i \pm \delta\lambda_i)\alpha_i = \boldsymbol{b}$$