

**Rule 2:** If  $M$  is of the form  $M = yf_1(x, y)$  and  $N$  is of the form  $N = xf_2(x, y)$ , and  $Mx - Ny \neq 0$ , then  $(Mx - Ny)^{-1}$  be an integrating factor (I.F.).

**Remark:** If  $Mx - Ny = 0$  i.e.  $\frac{M}{N} = \frac{y}{x}$ , then on substituting it in equation (1), we get  $\frac{y}{x} \cdot dx + dy = 0 \Rightarrow y \cdot dx + xdy = 0$

On integrating, we get the required solution  $xy = c$  (always in this case).

**Rule 3:** If the given equation  $Mdx + Ndy = 0$  is homogenous equation and  $Mx + Ny \neq 0$ , then  $(Mx + Ny)^{-1}$  is an I. F.

**Remark:** If  $Mx + Ny = 0$  i.e.,  $\frac{M}{N} = -\frac{y}{x}$ , then on substituting it in equation (1), we get  $-\frac{y}{x} dx + dy = 0 \Rightarrow \frac{dx}{x} = \frac{dy}{y}$

On integrating, we get the required solution  $x = cy$  (always in this case).

**Rule 4:** If  $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) / N$  is a function of  $x$  alone, say  $f(x)$ , then I.F. is equal to  $e^{\int f(x) dx}$ .

**Rule 5:** If  $\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) / M$  is a function of  $y$  alone, say  $f(y)$ , then I.F. is equal to  $e^{\int f(y) dy}$ .

**Rule 6:** If the equation  $\frac{dy}{dx} = f(x, y)$  is of the form  $x^a y^b [My dx + Nx dy] + x^r y^s [pydx + qxdy] = 0$ , where  $a, b, M, N, r, s, p$  and  $q$  are all constants, then I.F. =  $x^h y^k$ , where  $h$  and  $k$  are chosen such that after multiplying the given differential equation by I.F. it becomes exact. This exact differential equation can be solved by the above described method.

**Example 10:** Solve  $(x^2 - ay)dx = (ax - y^2)dy$ .

**Solution:** Given equation can be written as

$$(x^2 - ay)dx + (y^2 - ax)dy = 0 \quad \dots(1)$$

### 1.2.6 Type 6. Standard Linear Differential Equations

A differential equation of the form  $\frac{dy}{dx} + Py = Q$ , where  $P$  and  $Q$  are the functions of  $x$  alone, is called a linear differential equation.

**Solution of linear equation.** To solve such type of differential equation we multiply both sides by  $I.F. = e^{\int P dx}$

$$\text{We have } e^{\int P dx} \cdot dy + e^{\int P dx} \cdot Py dx = e^{\int P dx} Q dx$$

Hence on integrating both sides, we get

$$y e^{\int P dx} = \int \left[ Q \cdot e^{\int P dx} \right] dx + C$$

which is the required solution of the given linear differential equation.

**Example 18:** Solve  $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ .

**Solution:** We can write  $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$ , which is linear differential equation.

$$\text{Here } P = \frac{2x}{1+x^2}, Q = \frac{4x^2}{1+x^2}.$$

$$\text{Hence, integrating factor (I.F.)} = e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = 1+x^2.$$

Hence, the solution is given by  $I.F. \times y = \int I.F. \times Q dx + c$ , which gives

$$\begin{aligned} (1+x^2)y &= \int (1+x^2) \frac{4x^2}{(1+x^2)} dx + c \\ \Rightarrow (1+x^2)y &= \frac{4x^3}{3} + c. \end{aligned}$$

**Example 19:** Solve  $(1+y^2)dx = (\tan^{-1}y - x)dy$ .

**Solution:** We can write  $\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$ , which is linear equation in  $x$ .

$$\text{Thus } I.F. = e^{\int \frac{dy}{1+y^2}} \cdot dy = e^{\tan^{-1}y}.$$

$$\text{and the solution is } e^{\tan^{-1}y} x = \int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y} dy + c$$

Let  $\log y = t$  and  $\frac{1}{y} \frac{dy}{dx} = \frac{dt}{dx}$ , then (2) becomes  $\frac{dt}{dx} + \frac{t}{x} = e^x \dots(3)$

Now  $I.F. = e^{\int \frac{1}{x} dx} = e^{\log x} = x$ .

Hence, the solution is  $x \cdot \log y = \int x e^x dx + c$

$$\Rightarrow x \log y = x e^x - e^x + c.$$

**Example 24:** Solve  $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x) e^x \sec y$ . ...(1)

**Solution:** On dividing by  $\sec y$ , we get  $\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x) e^x \dots(2)$

Let  $\sin y = t \Rightarrow \cos y \frac{dy}{dx} = \frac{dt}{dx}$ , then (2) becomes

$$\frac{dt}{dx} - \frac{t}{1+x} = (1+x) e^x.$$

Now  $I.F. = e^{\int -\frac{1}{1+x} dx} = e^{-\log(1+x)} = \frac{1}{1+x}$ .

Hence, the solution is  $\frac{1}{1+x} \sin y = \int \frac{1}{1+x} (1+x) e^x dx + c$

$$\Rightarrow \frac{\sin y}{1+x} = e^x + c.$$

**Example 25:** Solve  $\frac{x dx + y dy}{x dy - y dx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$  ...(1)

**Solution:** Let  $x = r \cos \theta$ ,  $y = r \sin \theta$ , then we have

$$\frac{\partial x}{\partial r} = \cos \theta, \frac{\partial x}{\partial \theta} = -r \sin \theta, \frac{\partial y}{\partial r} = \sin \theta, \frac{\partial y}{\partial \theta} = r \cos \theta$$

By advanced calculus  $dx dy = J dr d\theta$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr d\theta$$

$$= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} dr d\theta$$

$$= r(\cos^2 \theta + \sin^2 \theta) dr d\theta = r dr d\theta.$$

Steps for finding the orthogonal trajectory for cartesian curves are as follows:

**Step 1.** Differentiate given family of curve  $f(x, y, c) = 0$  with respect to  $x$ , where  $c$  is parameter.

**Step 2.** Find the differential equation of the curve by eliminating parameter  $c$  between the equation of the given family of curves and the equation obtained in step 1.

**Step 3.** Replace  $\frac{dy}{dx}$  by  $-\frac{dx}{dy}$  to obtain the differential equation of the orthogonal trajectories.

**Step 4.** Solve this new differential equation to obtain the equation of orthogonal trajectories.

**(ii) Polar coordinates:** If  $\psi$  is the angle from the polar radius to the tangent, then  $\tan \psi = \frac{rd\theta}{dr}$ , as shown in Fig. 1.2.

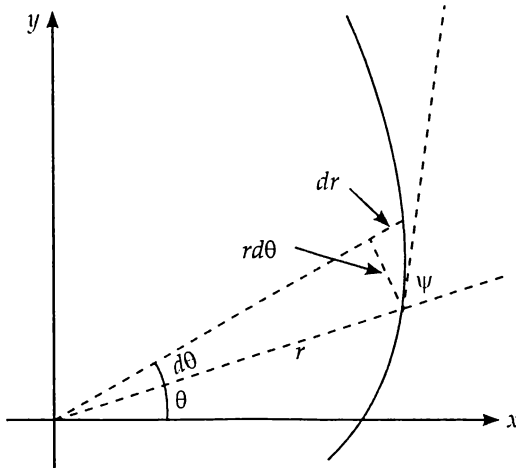


Fig. 1.2

Steps for finding the orthogonal trajectory for polar curves are as follows:

**Step 1.** Differentiate the polar equation of the family of curves  $f(r, \theta, \alpha) = 0$ , where  $\alpha$  being the parameter with respect to  $\theta$ .

**Step 2.** Find the differential equation of the curve by eliminating parameter  $\alpha$  between the equation of the given family of curves and the equation obtained in step 1. Find  $\frac{rd\theta}{dr}$ .

**Step 3.** Replace expression  $\frac{rd\theta}{dr}$  in the differential equation of the given family by its negative reciprocal  $-\frac{dr}{rd\theta}$  to obtain the differential equations of the orthogonal trajectories.

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