Rule 2: If M is of the form $M = yf_1(x.y)$ and N is of the form $N = xf_2(x.y)$, and $Mx - Ny \neq 0$, then $(Mx-Ny)^{-1}$ be an integrating factor (I.F.).

Remark: If $Mx - Ny = 0$ *i.e.* $\frac{M}{N} = \frac{y}{x}$, then on substituting it in equation (1), we get $\frac{y}{x}.dx + dy = 0 \Rightarrow y.dx + xdy = 0$

On integrating, we get the required solution $xy = c$ *(always in this case).*

Rule 3: If the given equation $Mdx+Ndy=0$ is homogenous equation and $Mx + Ny \neq 0$, then $(Mx+Ny)^{-1}$ is an I. F.

Remark: If $Mx + Ny = 0$ *i.e.,* $\frac{M}{N} = -\frac{y}{x}$, then on substituting it in equation (1), we get $-\frac{y}{x}dx + dy = 0 \Rightarrow \frac{dx}{x} = \frac{dy}{y}$

On integrating, we get the required solution $x = cy$ *(always in this case).*

Rule 4: If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\Big/N$ is a function of x alone, say $f(x)$, then I.F. is equal to $e^{\int f(x)dx}$.

Rule 5: If $\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right)\Big/M$ is a function of y alone, say $f(y)$, then I.F. is equal to $e^{\int f(y)dy}$.

Rule 6: If the equation $\frac{dy}{dx} = f(x, y)$ is of the form $x^a y^b\,[My\,dx + Nx\,dy] + x^r y^s\,[pydx + qxdy] = 0$, where a, b, M, N, r, s, p and q are all constants, then $I.F. = x^h y^k$, where h and k are chosen such that after multiplying the given differential equation by I.F. it becomes exact. This exact differential equation can be solved by the above described method.

Example 10: Solve $(x^2 - ay)dx = (ax - y^2)dy$.

Solution: Given equation can be written as

$$(x^2 - ay)dx + (y^2 - ax)dy = 0 \qquad ...(1)$$

1.2.6 Type 6. Standard Linear Differential Equations

A differential equation of the form $\frac{dy}{dx}+Py=Q$, where P and Q are the functions of x alone, is called a linear differential equation.

Solution of linear equation. To solve such type of differential equation we multiply both sides by $I.F. = e^{\int P dx}$

We have $e^{\int Pdx}.dy + e^{\int Pdx}.Py\,dx = e^{\int Pdx}Q\,dx$

Hence on integrating both sides, we get

$$ye^{\int Pdx} = \int\left[Q.e^{\int Pdx}\right]dx + C$$

which is the required solution of the given linear differential equation.

Example 18: Solve $(1+x^2)\frac{dy}{dx}+2xy-4x^2=0$.

Solution: We can write $\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{4x^2}{1+x^2}$, which is linear differential equation.

Here $P=\frac{2x}{1+x^2}$, $Q=\frac{4x^2}{1+x^2}$.

Hence, integrating factor $(I.F.) = e^{\int Pdx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2$.

Hence, the solution is given by $I.F.\times y=\int I.F.\times Q dx + c$, which gives

$$(1+x^2)y=\int(1+x^2)\frac{4x^2}{(1+x^2)}dx+c$$

$$\Rightarrow \quad (1+x^2)y=\frac{4x^3}{3}+c.$$

Example 19: Solve $(1+y^2)dx=(\tan^{-1}y-x)dy$.

Solution: We can write $\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$, which is linear equation in x.

Thus $I.F. = e^{\int\frac{dy}{1+y^2}}.dy = e^{\tan^{-1}y}$.

and the solution is $e^{\tan^{-1}y}x=\int\frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}dy+c$

Let $\log y = t$ and $\frac{1}{y}\frac{dy}{dx} = \frac{dt}{dx}$, then (2) becomes $\frac{dt}{dx} + \frac{t}{x} = e^x$ ...(3)

Now $I.F. = e^{\int \frac{1}{x} dx} = e^{\log x} = x.$

Hence, the solution is $x.\log y = \int xe^x dx + c$

$\Rightarrow \quad x \log y = xe^x - e^x + c.$

Example 24: Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)\, e^x \sec y.$ ...(1)

Solution: On dividing by sec y, we get $\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1-x)e^x$ ...(2)

Let $\sin y = t \Rightarrow \cos y \frac{dy}{dx} = \frac{dt}{dx}$, then (2) becomes

$\therefore \frac{dt}{dx} - \frac{t}{1+x} = (1+x)\, e^x.$

Now $I.F. = e^{\int -\frac{1}{1+x} dx} = e^{-\log(1+x)} = \frac{1}{1+x}.$

Hence, the solution is $\frac{1}{1+x} \sin y = \int \frac{1}{1+x}(1+x)e^x \, dx + c$

$\Rightarrow \quad \frac{\sin y}{1+x} = e^x + c.$

Example 25: Solve $\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$ ...(1)

Solution: Let $x = r\cos\theta$, $y = r\sin\theta$, then we have

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

By advanced calculus $dx\,dy = J\,dr\,d\theta$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta$$

$$= r(\cos^2\theta + \sin^2\theta)\,dr\,d\theta = r\,dr\,d\theta.$$

Steps for finding the orthogonal trajectory for cartesian curves are as follows:

Step 1. Differentiate given family of curve $f(x, y, c) = 0$ with respect to x, where c is parameter.

Step 2. Find the differential equation of the curve by eliminating parameter c between the equation of the given family of curves and the equation obtained in step 1.

Step 3. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to obtain the differential equation of the orthogonal trajectories.

Step 4. Solve this new differential equation to obtain the equation of orthogonal trajectories.

(ii) Polar coordinates: If ψ is the angle from the polar radius to the tangent, then $\tan\psi = \frac{rd\theta}{dr}$, as shown in Fig. 1.2.

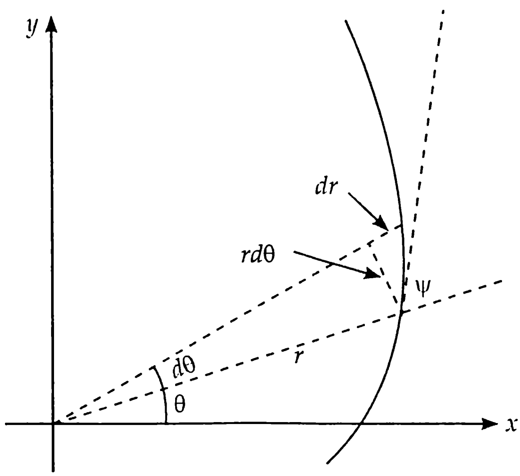

Fig. 1.2

Steps for finding the orthogonal trajectory for polar curves are as follows:

Step 1. Differentiate the polar equation of the family of curves $f(r, \theta, \alpha) = 0$, where α being the parameter with respect to θ.

Step 2. Find the differential equation of the curve by eliminating parameter α between the equation of the given family of curves and the equation obtained in step 1. Find $\frac{rd\theta}{dr}$.

Step 3. Replace expression $\frac{rd\theta}{dr}$ in the differential equation of the given family by its negative reciprocal $-\frac{dr}{rd\theta}$ to obtain the differential equations of the orthogonal trajectories.

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