CHAPTER-1

Partial Fractions

1.1 INTRODUCTION

It is a well known fact that the quotient of any two polynomials is known as a rational function. We also know the method of combining number of fractions by taking LCM (Least Common Multiple) of denominator of the fractions. The converse of this *i.e.* splitting up of a given fractions into a number of simpler fractions is called partial fractions. In this chapter we shall discuss the method of resolving a given function into partial fractions which will help us to integrate any given rational functions.

1.2 BASIC DEFINITIONS

Before discussing the partial fraction, let us recall the following definitions.

1. Polynomial. An algebraic expression of the form $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ $(a_n \neq 0)$ where $a_0, a_1, a_2, \ldots, a_n$ all are real constants and x is an unknown variable, is called a polynomial in x of degree n.

For example:

(i) $f(x) = x^3 + 3x^2 + 5x + 6$ is a polynomial of degree 3.

(ii) $f(x) = x^5 + 5x^4 + 6x^3 + 7x + 8$ is a polynomial of degree 5.

Remarks

- The expression $a_0, a_1x, a_2x^2, \ldots, a_nx^n$ are called terms of the polynomial.
- The real numbers $a_0, a_1, \ldots, a_n$ are called the coefficients of the polynomial.
- In a polynomial, if powers of x are either in an increasing or decreasing order, then polynomial is said to be in standard form.
- The highest power of the variable in a polynomial is called the degree of the polynomial.

2. Rational fraction. The quotient $\frac{P(x)}{Q(x)}$ of two polynomials $P(x)$ and $Q(x)$, where $Q(x) \neq 0$ is called a rational fraction.

For example:

Let $P(x) = x^3 + 3x^2 + 2x + 5$ and $Q(x) = x^4 + 3x^3 + 2x^2 + 3x + 6$

Then $\frac{P(x)}{Q(x)} = \frac{x^3 + 3x^2 + 2x + 5}{x^4 + 3x^3 + 2x^2 + 3x + 6}$ is a rational fraction.

3. Proper and Improper Fractions. The rational fraction $\frac{P(x)}{Q(x)}$ is called a proper fraction, if the degree of the numerator $P(x)$ is less than the degree of the denominator $Q(x)$. On the other hand, if the degree of numerator $P(x)$ is greater than or equal to the degree of the denominator $Q(x)$, then it is called improper fractions.

For example:

(i) $f(x)=\dfrac{x^2+2x+3}{x^3+3x^2+5x+6}$ is a proper fraction.

(ii) $f(x)=\dfrac{6x^3+3x^2+5x+6}{2x^2+3x+5}$ is an improper fraction.

1.3 PARTIAL FRACTIONS

Any proper rational fraction $\dfrac{f(x)}{g(x)}$ can be expressd as the sum of the rational fractions, each having a simple factor of $g(x)$. Each such fraction is called a partial fraction.

REMARK

- The process of obtaining partial fraction is called the decomposition or resolution of $\dfrac{f(x)}{g(x)}$ into partial fraction.

  For example: We have $\dfrac{2x}{x^2-1}=\dfrac{1}{x-1}+\dfrac{1}{x+1}$

  Then $\dfrac{1}{x-1}$ and $\dfrac{1}{x+1}$ are the partial fractions of the fraction $\dfrac{2x}{x^2-1}$.

FACTS : TO THE POINT

- The word 'Poly' means 'many', so polynomial means an algebraic expression consisting of many terms involving powers of the variable.
- A polynomial having only one term is known as **monomial**.
- A polynomial having two terms is known as **binomial**.
- A polynomial having all its coefficients zero is called zero polynomial.
- No degree is assigned to a zero polynomial *i.e.* the degree of a zero polynomial is not defined.
- The polynomial of degree one is called **linear polynomial**.
- The polynomial of degree two is called quadratic polynomial.
- The polynomial of degree three is called cubic polynomial.

1.4 METHODS OF RESOLUTION INTO PARTIAL FRACTIONS

The decomposition of $\dfrac{f(x)}{g(x)}$ into fraction depends on the nature of the factors of denominator. Here we have the following types.

TYPE 1. WHEN DENOMINATOR g(x) IS EXPRESSIBLE AS THE PRODUCT OF LINEAR NON-REPEATED FACTORS

If $ax + b$ is any linear non-repeated factor in the denominator then there corresponds a partial fraction of the form $\dfrac{1}{ax+b}$.

WORKING PROCEDURE

If $g(x) = (x - a_1)(x - a_2)\ldots(x - a_n)$. Then we assume that

$$\frac{f(x)}{g(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\ldots+\frac{A_n}{x-a_n}$$

where $A_1, A_2, \ldots, A_n$ are constants and can be determined by comparing the coefficients of various powers of x or by substituting $x = a_1, a_2, \ldots, a_n$ successively in the LHS and RHS after simplifications in the numerator.

REMARK

- If the given fraction is improper *i.e.* degree of numerator is greater than the degree of denominator. Then first dividing out the numerator by denominator to express the given improper fraction as the sum of polynomial and a proper fraction.

SOLVED EXAMPLES

EXAMPLE 1. ***Resolve $\frac{3x-1}{x^2-1}$ into partial fractions.***

SOLUTION. Clearly, the given fraction $\frac{3x-1}{x^2-1}$ is a proper fraction.

Also, $x^2 - 1 = (x - 1)(x + 1)$

i.e. denominator has linear non-repeated factors.

Therefore, we can write

$$\frac{3x-1}{x^2-1} = \frac{3x-1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$\Rightarrow \quad \frac{3x-1}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)} \quad \text{(By taking LCM)}$$

$$\Rightarrow \quad 3x - 1 = A(x + 1) + B(x - 1) = (A + B)x + (A - B) \quad \text{...(1)}$$

To find the values of A and B we have the following two methods

Method 1. Comparing the coefficients of like powers of x on both sides of (1), we get

$$A + B = 3 \text{ and } A - B = -1$$

On solving we get

$$A = 1 \text{ and } B = 2$$

Hence, $\frac{3x-1}{x^2-1} = \frac{1}{x-1} + \frac{2}{x+1}$

Method 2. We can find the values of A and B by giving some specific values to x in (1)

Let us put $x = 1$ in (1). Then

$$3(1) - 1 = A(1 + 1) + B(1 - 1) \Rightarrow 2A = 2 \text{ i.e. } A = 1$$

Similarly, put $x = 2$ in (1), we get

$$3(2) - 1 = A(2 + 1) + B(2 - 1)$$

$\Rightarrow \quad 3A + B = 5$

$\Rightarrow \quad B = 5 - 3A = 5 - 3(1) \quad (\because A = 1)$

$\Rightarrow \quad B = 2$

Hence, $\frac{3x-1}{x^2-1} = \frac{1}{x-1} + \frac{2}{x+1}$.

REMARKS

- It is always convenient to use those values of x which makes linear factors in the denominator zero.
- This short cut method is to be used only when the denominator has only linear non-repeated factors.

EXAMPLE 2. ***Resolve $\frac{x}{(x+1)(x-1)(x+2)}$ into partial fractions.***

SOLUTION. Clearly, the given fraction is a proper fraction. Therefore, we can write

$$\frac{x}{(x+1)(x-1)(x+2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$\Rightarrow \quad x = A(x - 1)(x + 2) + B(x + 1)(x + 2) + C(x + 1)(x - 1) \quad \text{...(1)}$$

Put $x = 1$ in (1), we get

$$1 = 6B \text{ i.e. } B = 1/6$$

Put $x = -1$ in (1), we get $A = 1/2$
Put $x = 2$ in (1), we get $C = -2/3$

Hence, $\frac{x}{(x+1)(x-1)(x+2)} = \frac{1}{2(x+1)} + \frac{1}{6(x-1)} - \frac{2}{3(x+2)}$

EXAMPLE 3. ***Resolve*** $\frac{x^2}{(x^2+1)(x^2+4)}$ ***into partial fractions.***

SOLUTION. Let us take $x^2 = t$, then we can write

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{t}{(t+1)(t+4)}$$

Clearly, it is a proper fraction. Therefore, we can write

$$\frac{t}{(t+1)(t+4)} = \frac{A}{(t+1)} + \frac{B}{(t+4)} = \frac{A(t+4)+B(t+1)}{(t+1)(t+4)}$$

$\Rightarrow$ $t = A(t + 4) + B(t + 1)$ ...(1)

Put $t = -1$ in (1), we get $-1 = 3A$ $\Rightarrow$ $A = -1/3$
Put $t = -4$ in (1), we get $-4 = -3B$ $\Rightarrow$ $B = 4/3$

Thus, $\frac{t}{(t+1)(t+4)} = \frac{-1}{3(t+1)} + \frac{4}{3(t+4)}$

$\Rightarrow$ $\frac{x^2}{(x^2+1)(x^2+4)} = \frac{-1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$ $(\because x^2 = t)$

EXAMPLE 4. ***Resolve*** $\frac{x^3-6x^2+10x-2}{x^2-5x+6}$ ***into partial fractions.***

SOLUTION. Clearly, the degree of numerator is greater than the degree of denominator, so it is an improper fraction. So, firstly we shall divided the numerator by denominator to make it proper fraction.

So, $\frac{x^3-6x^2+10x-2}{x^2-5x+6} = (x+1) + \frac{(4-x)}{x^2-5x+6}$

Now we have to resolve $\frac{4-x}{x^2-5x+6}$ into partial fraction.

Now, $\frac{4-x}{x^2-5x+6} = \frac{4-x}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$

$\Rightarrow$ $4-x = A(x-3)+B(x-2)$ ...(1)

Put $x = 3$ in (1), we get $B = 1$
Put $x = 2$ in (1), we get $A = -2$

So, $\frac{4-x}{x^2-5x+6} = \frac{-2}{x-2} + \frac{1}{x-3}$

Hence, $\frac{x^3-6x^2+10x-2}{x^2-5x+6} = (x-1) - \frac{2}{x-2} + \frac{1}{x-3}$

Division Scheme

$$\begin{array}{r|l} & x-1 \\ x^2-5x+6 & x^3-6x^2+10x-2 \\ & \underset{-}{x^3}\ \underset{+}{-5x^2}\ \underset{-}{+6x} \\ \hline & -x^2+4x-2 \\ & \underset{+}{-x^2}\ \underset{-}{+5x}\ \underset{+}{-6} \\ \hline & -x+4 \end{array}$$

TYPE 2. IF (ax + b) IS ANY LINEAR FACTOR REPEATED r (r ∈ N) TIMES IN THE DENOMINATOR

In this case corresponding partial fraction will be of the following form

$\frac{A}{ax+b}, \frac{B}{(ax+b)^2}, \ldots, \frac{C}{(ax+b)^3}, \ldots r$ times

More specific, when denominator $g(x)$ is expressible as the product of linear factors such that some of them are repeating *i.e.*

$$g(x) = (x-a)^r(x-a_1)(x-a_2)\ \ldots\ (x-a_k)$$

Then we assume that

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_r}{(x-a)^r} + \frac{B_1}{(x-a_2)} + \frac{B_2}{(x-a_2)} + \ldots + \frac{B_k}{(x-a_k)}$$

SOLVED EXAMPLES

EXAMPLE 1. ***Resolve $\frac{(2x+1)}{(x+2)(x-3)^2}$ into partial fractions.***

SOLUTION. Clearly, the given partial fraction is a proper fraction. Let the partial fraction corresponding to the factor $(x + 2)$ be $\frac{A}{x+2}$ and the partial fractions corresponding to the factors $(x-3)^2$ be $\frac{B}{(x-3)}$ and $\frac{C}{(x-3)^2}$. Thus, we can write

$$\frac{2x+1}{(x+2)(x-3)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

$\Rightarrow$ $2x + 1 = A(x-3)^2 + B(x+2)(x-3) + C(x+2)$ ...(1)

Putting $x = -2$ in (1), we get

$$-3 = A(-5)^2 + B\cdot 0 + C\cdot 0 \quad \Rightarrow \quad A = \frac{-3}{25}$$

Now, putting $x = 3$ in (1), we get

$$7 = A\cdot 0 + B\cdot 0 + C\cdot 5 \quad \Rightarrow \quad C = \frac{7}{5}$$

Further comparing the coefficients of x^2 in both sides of (1), we get

$$0 = A + B$$

$$\Rightarrow \quad B = -A = -\left(\frac{-3}{25}\right) = \frac{3}{25}$$

Hence, $\frac{2x+1}{(x+2)(x-3)^2} = \frac{-3}{25(x+2)} + \frac{3}{25(x-3)} + \frac{7}{5(x-3)^2}$

EXAMPLE 2. ***Resolve $\frac{2x^2}{(x-1)^3(x+1)}$ into partial fractions.***

SOLUTION. Proceed same as in previous example, we can write

$$\frac{2x^2}{(x-1)^3(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x+1)}$$

$\Rightarrow$ $2x^2 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3$ ...(1)

Putting $x = 1$ in both the sides we get

$C = 1$ ...(2)

Now, equating the coefficients of x^3, x^2, x and constant terms on both the sides of (1), we get

$A + D = 0$ ...(3)

$-A + B - 3D = 2$ ...(4)

$-A + C + 3D = 0$ ...(5)

$A - B + C - D = 0$ ...(6)

On solving (3) to (6) and using (2), we get

$$A=\frac{1}{4}, B=\frac{3}{2} \text{ and } D=\frac{-1}{4}$$

Hence, $\dfrac{2x^2}{(x-1)^3(x+1)}=\dfrac{1}{4(x-1)}+\dfrac{3}{2(x-1)^2}+\dfrac{1}{(x-1)^3}-\dfrac{1}{4(x+1)}$

EXAMPLE 3. ***Resolve $\dfrac{x}{(x+1)^3(x-2)^2}$ into partial fractions.***

SOLUTION. Let us write

$$\frac{x}{(x+1)^3(x-2)^2}=\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}+\frac{D}{(x-2)}+\frac{E}{(x-2)^2}$$

$\Rightarrow$ $x=A(x+1)^2(x-2)^2+B(x+1)(x-2)^2+C(x-2)^2+D(x-2)(x+1)^3+E(x+1)^3$ ...(1)

Putting $x = -1$ in (1), we get

$C = -1/9$ ...(2)

Now, put $x = 2$ in (1), we get

$E = 2/27$ ...(3)

Further, equating the coefficients of x^4, x^3 and constants on both the sides of (1), we get

$0 = A + D$ ...(4)

$0 = -2A + B + D + E$ ...(5)

and $0 = A + 4B + 4C - 2D + E$ ...(6)

On solving (4) to (6) with the help of (2) and (3), we get

$$A=\frac{1}{27}, B=\frac{1}{27}, D=-\frac{1}{27}$$

Hence, we can write

$$\frac{x}{(x+1)^3(x-2)^2}=\frac{1}{27(x+1)}+\frac{1}{27(x+1)^2}-\frac{1}{9(x+1)^3}-\frac{1}{27(x-2)}+\frac{2}{27(x-2)^2}$$

TYPE 3. WHEN $ax^2 + bx + c$ IS AN IRREDUCIBLE QUADRATIC FACTOR IN THE DENOMINATOR THEN THERE CORRESPONDS A PARTIAL FRACTION OF THE FORM $\dfrac{Ax+B}{ax^2+bx+c}$

This method will be clear by the following examples.

SOLVED EXAMPLES

EXAMPLE 1. ***Resolve $\dfrac{x}{(x^2+1)(x+1)}$ into partial fractions.***

SOLUTION. Clearly, $(x^2 + 1)$ is an irreducible quadratic factor in the denominator. Therefore, we can write

$$\frac{x}{(x^2+1)(x+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1}$$

$\Rightarrow$ $x = (Ax + B)(x + 1) + C(x^2 + 1)$ ...(1)

We have to find the values of A, B and C

Putting $x = -1$ in (1), we get

$-1 = (-A + B)\cdot 0 + C\cdot 2$

$\Rightarrow$ $C = -1/2$ ...(2)

Now, comparing the coefficients of like powers of x in (1), we get

$$0 = A + C \text{ and } A + B = 1$$

$\Rightarrow$ $A = -C = -(-1/2) = 1/2$

and $B = 1 - A = 1 - 1/2 = 1/2$

Hence, $\dfrac{x}{(x^2+1)(x+1)} = \dfrac{x+1}{2(x^2+1)} - \dfrac{1}{2(x+1)}$

EXAMPLE 2. ***Resolve $\dfrac{1}{1+x^3}$ into partial fractions.***

SOLUTION. We know that

$$1+x^3 = (1+x)(1+x^2-x)$$

$\therefore$ $\dfrac{1}{1+x^3} = \dfrac{1}{(1+x)(1+x^2-x)}$

Clearly, $(1 + x^2 - x)$ is an irreducible quadratic factor. So, we can write

$$\frac{1}{1+x^3} = \frac{1}{(1+x)(1+x^2-x)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2-x}$$

$\Rightarrow$ $1 = A(1+x^2-x) + (Bx+C)(1+x)$ ...(1)

Put x = –1 in (1), we get A = 1/3

Now, equating the coefficients of x^2 and constant terms on both the sides of (1), we get

$0 = A + B \Rightarrow B = -A = -(1/3) = -1/3$

and $1 = A + C \Rightarrow C = -A = -1/3$

Hence, $\dfrac{1}{1+x^3} = \dfrac{1}{3(x+1)} - \dfrac{(x+1)}{3(1+x^2-x)}$

EXAMPLE 3. ***Resolve $\dfrac{2x+1}{(x^2+2)(x^2+3)}$ into partial fractions.***

SOLUTION. Here, both the factors in the denominator are irreducible and quadratic. So, we can write

$$\frac{2x+1}{(x^2+2)(x^2+3)} = \frac{Ax+B}{(x^2+2)} + \frac{Cx+D}{(x^2+3)}$$

$\Rightarrow$ $2x+1 = (Ax+B)(x^2+3) + (Cx+D)(x^2+2)$ ...(1)

We have to find the values of A, B, C and D.

Equating the coefficients of x^3, x^2, x and constant terms on both the sides on (1), we get

$0 = A + C$ ...(2)

$0 = B + D$ ...(3)

$2 = 3A + 2C$ ...(4)

$1 = 3B + 2D$ ...(5)

From (2), $A = -C$. Putting this value in (4), we get

$$2 = 3(-C) + 2C$$

$\Rightarrow$ $C = -2$

and thus $A = 2$

Similarly, using (3) in (5), we get

$$B = 1 \text{ and } D = -1$$

Hence, we can write

$$\frac{2x+1}{(x^2+2)(x^2+3)} = \frac{2x+1}{(x^2+2)} - \frac{2x+1}{(x^2+3)}$$

TYPE 4. WHEN $ax^2 + bx + c$ IS ANY IRREDUCIBLE FACTOR REPEATED r ($r \in N$) TIMES IN THE DENOMINATOR THEN THERE CORRESPONDS PARTIAL FRACTIONS OF THE FORM $\frac{Ax+B}{ax^2+bx+c}, \frac{Cx+D}{(ax^2+bx+c)^2}, \ldots$ r TIMES

SOLVED EXAMPLES

EXAMPLE 1. ***Resolve $\frac{2x^3}{(x^2+1)^2}$ into partial fractions.***

SOLUTION. Clearly, the factor $(x^2 + 1)$ is irreducible, quadratic and repeated two times. So, we can write

$$\frac{2x^3}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{(Cx+D)}{(x^2+1)^2}$$

$$\Rightarrow \quad 2x^3 = (Ax+B)(x^2+1) + (Cx+D)$$

$$\Rightarrow \quad 2x^3 = Ax^3 + Bx^2 + (A+C)x + B + D \quad \ldots(1)$$

Comparing the coefficients of x^3, x^2, x and constant terms in (1) we get

$$A = 2, B = 0, A + C = 0, B + D = 0$$

$$\Rightarrow \quad A = 2, B = 0, C = -2, D = 0$$

Hence, we can write

$$\frac{2x^3}{(x^2+1)^2} = \frac{2x+0}{x^2+1} + \frac{-2x+0}{(x^2+1)^2}$$

$$= \frac{2x}{x^2+1} - \frac{2x}{(x^2+1)^2}$$

EXAMPLE 2. ***Resolve $\frac{2x-3}{(x-1)(x^2+1)^2}$ into partial fractions.***

SOLUTION. Clearly, $(x^2 + 1)$ is an irreducible quadratic factor in the denominator and repeated two times. So, we can write

$$\frac{2x-3}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\Rightarrow \quad 2x-3 = A(x^2+1)^2 + (Bx+C)(x^2+1)(x-1) + (Dx+E)(x-1) \quad \ldots(1)$$

Equating the coefficients of like powers of x and constant terms on both sides of (1), we get

$$\left.\begin{aligned} A+B &= 0 \\ C-B &= 0 \\ 2A+B-C+D &= 0 \\ C+E-B-D &= 0 \\ A-C-E &= -3 \end{aligned}\right] \quad \ldots(2)$$

Also, put $x = 1$ in (1), we get

$$A = -1/4 \quad \ldots(3)$$

On solving (2) and using (3), we get $B = \frac{1}{4}, C = \frac{1}{4}, D = \frac{1}{2}, E = \frac{5}{2}$

Hence, $\frac{2x-3}{(x-1)(x^2+1)^2} = -\frac{1}{4(x-1)} + \frac{(x+1)}{4(x^2+1)} + \frac{(x+5)}{2(x^2+1)^2}$

EXAMPLE 3. ***Resolve*** $\dfrac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3}$ ***into partial fraction.***

SOLUTION. We can write

$$\frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} + \frac{Ex+F}{(x^2+2)^3} \quad \ldots(1)$$

$$\Rightarrow \quad x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4$$
$$= (Ax+B)(x^2+2)^2 + (Cx+D)(x^2+2) + (Ex+F)$$
$$= Ax^5 + Bx^4 + (4A+C)x^3 + (4B+D)x^2 + (4A+2C+E)x + (4B+2D+F) \quad \ldots(2)$$

Comparing the coefficients of like powers of x on both the sides of (2), we get

$$A = 1, B = -1, 4A + C = 4, 4B + D = -4, 4A + 2C + E = 8, 4B + 2D + F = -4$$

On solving we get

$$A = 1, B = -1, C = 0, D = 0, E = 4, F = 0$$

Putting all these values in (1), we get

$$\frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3} = \frac{x-1}{(x^2+2)} + \frac{4x}{(x^2+2)^3}$$

1.5 APPLICATIONS OF PARTIAL FRACTIONS IN CHEMICAL KINETIC AND PHARMACOKINETICS

We know that Chemical kinetic is that branch of chemistry which deals with the rate of chemical reactions. We have the following applications of partial fractions in Chemical kinetics and Pharmacokinetics.

1.5.1 DIFFERENTIAL RATE EXPRESSION

In a second order kinetics, the differential rate expression is given by $r = \dfrac{dx}{dt} = c(a-x)(b-x)$

We can solve the above equation by using partial fraction.

For, $\dfrac{dx}{dt} = c(a-x)(b-x)$

On seperating the variables, we get

$$\frac{dx}{(a-x)(b-x)} = c\,dt \quad \ldots(1)$$

Now, we have to obtain the partial fraction of $\dfrac{1}{(a-x)(b-x)}$

Let us write

$$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x}$$

$$\Rightarrow \quad 1 = A(b-x) + B(a-x) \quad \ldots(2)$$

Putting $x - b = 0$ *i.e.* $x = b$ in (2), we get

$$1 = 0 + B(a-b)$$

$$\Rightarrow \quad B = \frac{1}{a-b}$$

Further, put $a - x = 0$ *i.e.* $x = a$ in (2) we get

$$1 = A(b-x) + B(a-x)$$

$$\Rightarrow \quad 1 = A(b-a) + 0$$

$$\Rightarrow \quad A = \frac{1}{b-a}$$

Thus,

$$\frac{1}{(a-x)(b-x)} = \frac{1}{(b-a)(a-x)} + \frac{1}{(a-b)(b-x)}$$

$$= \frac{1}{-(a-b)(a-x)} + \frac{1}{(a-b)(b-x)}$$

$$= \frac{1}{(a-b)(b-x)} - \frac{1}{(a-b)(a-x)} = \frac{1}{a-b}\left[\frac{1}{b-x} - \frac{1}{a-x}\right]$$

We can solve easily equation (1) by using this value of partial fraction.

1.5.2 WHEN THE DRUG ADMINISTERED INTRAVENOUSLY AT A CONSTANT RATE

If the drug is administered intravenously at a constant rate, the rate of change of any drug in the body is given by the equation $\frac{dx}{dt} = k_0 - kx$, where k_0 is the rate of drug infusion.

To solve it, let us take Laplace transforms on both the sides,

$$L\left\{\frac{dx}{dt}\right\} + kL(x) = k_0 L\{1\}$$

$$\Rightarrow \quad pL(x) - x(0) + kL\{x\} = k_0 \cdot \frac{1}{p}$$

$$\Rightarrow \quad (p+k)L\{x\} = \frac{k_0}{p}$$ ($x(0) = 0$, because, initially the drug in the body is zero)

$$\Rightarrow \quad \bar{x} = \frac{k_0}{p(p+k)} \qquad \ldots(1)$$

We have to apply method of partial fraction to solve the RHS of (1)

Let $$\frac{k_0}{p(p+k)} = \frac{A}{p} + \frac{B}{p+k}$$

$$\Rightarrow \quad k_0 = A(p+k) + Bp$$

Put $p = 0$, we get $kA = k_0 \Rightarrow \quad A = \frac{k_0}{k}$

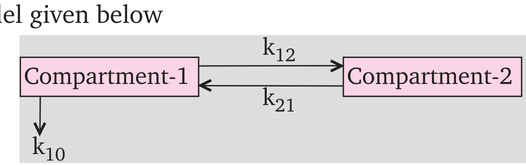
Put $p = -k$, we get $k_0 = -kB \quad \Rightarrow \quad B = -\frac{k_0}{k}$

Hence, $$\frac{k_0}{p(p+k)} = \frac{k_0}{kp} - \frac{k_0}{k(p+k)}$$

$$= \frac{k_0}{k}\left[\frac{1}{p} - \frac{1}{p+k}\right]$$

1.5.3 TWO COMPARTMENT MODEL FOR RAPID INTRAVENOUS INJECTION

The rapid interavenous injection of a drug that distribute in the body according to two compartment model system with elimination occuring from the central compartment is shown in the model given below



And the amount of drug in the central compartment a_{sc} is given by the equation

$$a_{sc} = x_0 \frac{(s+E_2)}{(s+\lambda_1)(s+\lambda_2)} \qquad \ldots(1)$$

To calculate concentration of drug in the central compartment, the RHS of (1) has to be resolved into partial fractions

For this let $\frac{s+E_2}{(s+\lambda_1)(s+\lambda_2)} = \frac{A}{s+\lambda_1} + \frac{B}{s+\lambda_2}$

$$\Rightarrow \quad s+E_2 = A(s+\lambda_2)+B(s+\lambda_1) \quad \ldots(2)$$

Put $s = -\lambda_2$ in (2), we get

$$E_2 - \lambda_2 = B(\lambda_1 - \lambda_2)$$

$$\Rightarrow \quad B = \frac{E_2-\lambda_2}{\lambda_1-\lambda_2}$$

Further, put $s_1 = -\lambda_1$ in (2), we get

$$E_2 - \lambda_1 = (\lambda_2 - \lambda_1)A$$

$$\Rightarrow \quad A = \frac{E_2-\lambda_1}{\lambda_2-\lambda_1}$$

Thus, $$\frac{s+E_2}{(s+\lambda_1)(s+\lambda_2)} = \frac{E_2-\lambda_1}{\lambda_2-\lambda_1}\cdot\frac{1}{s+\lambda_1} + \frac{E_2-\lambda_2}{\lambda_1-\lambda_2}\cdot\frac{1}{s+\lambda_2}$$

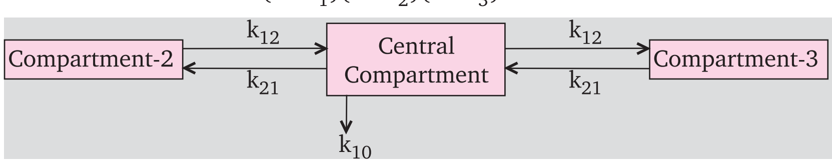
$$= \frac{1}{\lambda_2-\lambda_1}\left[\frac{E_2-\lambda_1}{s+\lambda_1} - \frac{E_2-\lambda_2}{s+\lambda_2}\right]$$

Hence, $a_{sc} = x_0\left[\frac{1}{\lambda_2-\lambda_1}\left(\frac{E_2-\lambda_1}{s+\lambda_1} - \frac{E_2-\lambda_2}{s+\lambda_2}\right)\right]$.

1.5.4 THREE COMPARTMENT MODEL FOR AMOUNT OF DRUG

The disposition function for the Central Compartment in three compartment model is given by the equation

$$a_{sc} = \frac{(s+E_2)(s+E_3)}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)}$$



Amount of drug in Central Compartment a_{sc} which is product of input and disposition is given by

$$a_{sc} = x_0\frac{(s+E_2)(s+E_3)}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)} \quad \ldots(1)$$

For the solution of (1) we have to resolve RHS of (1) into partial fractions.

Let $\frac{(s+E_2)(s+E_3)}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)} = \frac{A}{(s+\lambda_1)} + \frac{B}{(s+\lambda_2)} + \frac{C}{(s+\lambda_3)}$

$$\Rightarrow \quad (s+E_2)(s+E_3) = A(s+\lambda_2)(s+\lambda_3)+B(s+\lambda_1)(s+\lambda_3)+C(s+\lambda_1)(s+\lambda_2) \quad \ldots(2)$$

Putting $s = -\lambda_1$, we get

$$(E_2-\lambda_1)(E_3-\lambda_1) = A(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)$$

$$\Rightarrow \quad A = \frac{(E_2-\lambda_1)(E_3-\lambda_1)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}$$

Put $s = -\lambda_2$, we get

$$B = \frac{(E_2-\lambda_2)(E_3-\lambda_2)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}$$

and by putting $s = -\lambda_3$ and proceed same as above, we get

$$C = \frac{(E_2-\lambda_3)(E_3-\lambda_3)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$$

Hence,

$$x_0\frac{(s+E_2)(s+E_3)}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)} = x_0\left[\frac{1}{(s+\lambda_1)}\left(\frac{(E_2-\lambda_1)(E_3-\lambda_1)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\right)+\frac{1}{(s+\lambda_2)}\left(\frac{(E_2-\lambda_2)(E_3-\lambda_2)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}\right)\right.$$
$$\left.+\frac{1}{(s+\lambda_3)}\left(\frac{(E_2-\lambda_3)(E_3-\lambda_3)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right)\right]$$

1.5.5 FIRST ORDER ABSORPTION (INFUSION METHOD)

For a drug that enters the body by an apparent first order absorption process is eliminated by a first order process and distributes in the body is given by the equation

$$\frac{dx}{dt} = k_0x_0 - kx$$

Using Laplace transform technique (As in (1.5.2)) we can find

$$\bar{x} = \frac{k_0Fx_0}{(s+k)(s+k_0)} \qquad \ldots(1)$$

To find the value of x, we have to resolve the RHS of (1) into partial fractions.

Let $$\frac{1}{(p+k)(p+k_0)} = \frac{A}{p+k}+\frac{B}{p+k_0}$$

$$\Rightarrow \qquad 1 = A(p+k_0)+B(p+k) \qquad \ldots(2)$$

Put $p + k = 0$ *i.e.* $p = -k$ in (1), we get

$$A = \frac{1}{k_0-k}$$

Now, put $p + k_0 = 0$ *i.e.* $p = -k_0$ in (1), we get

$$B = \frac{1}{k-k_0}$$

Hence, $$\frac{1}{(p+k)(p+k_0)} = \frac{1}{(k_0-k)}\left[\frac{1}{p+k}-\frac{1}{p+k_0}\right]$$

EXERCISE 1.1

Decompose the following rational functions into partial fractions:

1. $\dfrac{3x-1}{x^2-1}$ **2.** $\dfrac{2x+3}{(x+1)(x^2+1)}$

3. $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}$

4. $\dfrac{x^2+5}{(x^2+x+1)(x^2-x+2)}$

5. $\dfrac{x^2+2}{(x^2+1)(x^2+4)}$ **6.** $\dfrac{12x^2-2x-9}{(4x^2-1)(x+3)}$

7. $\dfrac{x^4+x^3+2x^2+4x+1}{x(x+1)}$

8. $\dfrac{x^4}{x^4-1}$ **9.** $\dfrac{x}{1+x+x^2+x^3}$

ANSWERS

1. $\frac{1}{x+1}+\frac{2}{x+1}$
2. $\frac{1}{2(x+1)}+\frac{5-x}{2(x^2+1)}$
3. $(2x+3)+\frac{1}{x-1}+\frac{5}{3x+1}$
4. $\frac{x+2}{x^2+x+1}-\frac{x-1}{x^2-x+2}$
5. $\frac{1}{3(x^2+1)}+\frac{2}{3(x^2+4)}$
6. $-\frac{1}{2x-1}+\frac{1}{2x+1}+\frac{3}{x+3}$
7. $\frac{1}{x}+\frac{1}{x+1}+(x^2+2)$
8. $1+\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x^2+1)}$
9. $\frac{x+1}{2(x^2+1)}-\frac{1}{2(x+1)}$

OBJECTIVE EVALUATION

FILL IN THE BLANKS

1. A fraction of which the numerator is of lessor degree than its denominator is called ______________.
2. The reverse process of breaking up a single fraction into simpler fraction whose denominators are the factors of the denominator of the given fraction is called resolution of fraction into ______________.
3. Any improper fraction can be expressed as the sum of a polynomial and a ______________ fraction.
4. If $(ax + b)$ is any linear non-repeated factor in the denominator then there corresponds a partial fraction of the form ______________.

TRUE/FALSE (WRITE T FOR TRUE AND F FOR FALSE STATEMENT)

1. If deg.$f(x)$ < deg.g(x) then $\frac{f(x)}{g(x)}$ is a proper fraction. **(T/F)**
2. If deg.$f(x)$ > deg.g(x) then $\frac{f(x)}{g(x)}$ is an improper fraction. **(T/F)**
3. If $\frac{f(x)}{g(x)}$ is an imporper rational fraction then we divide $f(x)$ by $g(x)$ so as to write $\frac{f(x)}{g(x)}$ as the sum of a polynomial and a proper rational function. **(T/F)**
4. Any improper fraction can be expressed as the product of a polynomial and a proper fraction. **(T/F)**

MULTIPLE CHOICE QUESTIONS (CHOOSE THE MOST APPROPRIATE ONE)

1. If $\frac{x^2}{(x^2+1)(x^2+4)}=\frac{a}{x^2+1}+\frac{b}{x^2+4}$, then value of $a + b =$
   (a) 1 (b) 0
   (c) 2 (d) none of these
2. If $\frac{3-4x-5x^2-x^3}{(x+3)(x+2)}=-x+\frac{a}{x+3}+\frac{b}{x+2}$, then $a + b =$
   (a) 1 (b) 0
   (c) 2 (d) none of these
3. If $\frac{5x^2+8x-2}{(3x-2)(x^2-x+3)}=\frac{a}{3x-2}+\frac{bx+c}{x^2-x+3}$, then $a + b + c =$
   (a) 1 (b) 7
   (c) 8 (d) none of these
4. If $\frac{ax}{(x+2)(2x-3)}=\frac{2}{x+2}+\frac{3}{2x-3}$, then $a =$
   (a) 3 (b) 5
   (c) 7 (d) none of these

ANSWERS

FILL IN THE BLANKS

1. proper fraction **2.** partial fraction **3.** proper **4.** $\frac{1}{ax+b}$

TRUE/ FALSE

1. T **2.** T **3.** T **4.** F

MULTIPLE CHOICE QUESTIONS

1. (a) **2.** (c) **3.** (b) **4.** (c)