CHAPTER 1

Discrete Fourier Transform

1.1 Review of Signals and Systems

1.1.1 Signals

Any physical phenomenon that conveys or carries some information can be called a signal.

Examples: Music, speech, motion pictures, still photos, heart beat, etc.

Signal: *A physical quantity that varies with one or more independent variables is called a signal.*

The independent variables can be time, spatial coordinates, intensity of colours, pressure, temperature, etc. The most popular independent variable in signals is time and it is represented by the letter "t".

Amplitude: *The value of a signal at any specified value of the independent variable is called its amplitude.*

Waveform: *The sketch or plot of the amplitude of a signal as a function of the independent variable is called its waveform.*

Mathematically, any signal can be represented as a function of one or more independent variables.

Table 1.1: Examples of Signals

Basis for Classification	Type	Definition	Example
Number of sources	One-channel signals	Signals that are generated by a single source.	i) Record of room temperature. ii) Audio output of a monospeaker.
	Multi-channel signals	Signals that are generated by multiple sources.	i) Record of ECG at eight different places in a human body. ii) Audio output of two stereo speakers.
Number of dimensions	One-dimensional signals	Signal which is a function of a single independent variable.	i) Music, speech and heart beat which are function of a single independent variable, time. ii) $x_1(t) = 0.7t$.
	Multi-dimensional signals	Signal which is a function of two or more independent variables.	i) Photograph is two-dimensional (2D) signal. ii) Motion picture of a black and white TV is a three-dimensional (3D) signal.

Table 1.1: Continued...

Basis for Classification	Type	Definition	Example
Whether the dependent variable is continuous or discrete	Analog or continuous signal	Signal which is defined continuously for any value of the independent variable is called **analog signal**. When the independent variable of an analog signal is time, it is called **continuous time signal.**	Most of the signals encountered in science and engineering are analog.
	Discrete signal	Signal which is defined for discrete intervals of the independent variable is called **discrete signal.** When the independent variable of a discrete signal is time, it is called **discrete time signal.**	Sampled version of analog signal.

When a signal is defined continuously for any value of an independent variable, it is called an ***analog*** or ***continuous signal***. When the dependent variable of an analog signal is time, it is called a ***continuous time signal*** and it is denoted as "x(t)".

When a signal is defined for discrete intervals of an independent variable, it is called a ***discrete signal***. When the dependent variable of a discrete signal is time, it is called ***discrete time signal*** and it is denoted by "x(n)".

The quantized and coded version of the discrete time signals are called ***digital signals***. In digital signals the value of the signal for every discrete time "n" is represented in binary codes.

1.1.2 Continuous Time Signal

Continuous time signal: *A signal which is defined continuously for any value of the independent variable time "t" is called continuous time signal and it is denoted as "x(t)".*

Example: Sinusoidal signal, $x(t) = A \sin \Omega_0 t$,

where, $\Omega_0 = 2\pi F = 2\pi/T$

The continuous time signal is defined for every instant of the independent variable time and so the magnitude (or the value) of continuous time signal is continuous in the specified range of time. Here both the magnitude of the signal and the independent variable are continuous.

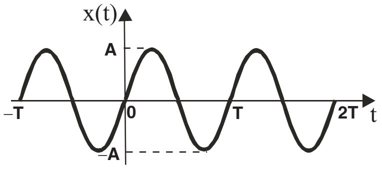


Fig 1.1: *Sinusoidal signal.*

1.1.3 Discrete Time Signal

Discrete signal: *It is a function of a discrete independent variable. The independent variable is divided into uniform intervals and each interval is represented by an integer. The letter "n" is used to denote the independent variable. The discrete or digital signal is denoted by x(n).*

The discrete signal is defined for every integer value of the independent variable "n". The magnitude (or value) of discrete signal can take any discrete value in the specified range.

Here both the value of the signal and the independent variable are discrete. The discrete signal can be represented by a one-dimensional array as shown in the following example.

Example:

$x(n) = \{ 2, 4, -1, 3, 3, 4 \}$

Here the discrete signal x(n) is defined for, $n = 0, 1, 2, 3, 4, 5$

$\therefore$ $x(0) = 2$; $x(1) = 4$; $x(2) = -1$; $x(3) = 3$; $x(4) = 3$; $x(5) = 4$

Discrete time signal: *When the independent variable is time t, the discrete signal is called discrete time signal. In discrete time signal, time is divided uniformly using the relation t = nT, where T is the sampling time period. The sampling time period is the inverse of sampling frequency. The discrete time signal is denoted by x(n) or x(nT).*

Discrete sequence: *Discrete signals have a sequence of numbers (or values) defined for integer values of the independent variable,and hence are also known as discrete sequence.*

In this book, the term sequence and signal are used synonymously. Also in this book, the discrete signal is referred as discrete time signal.

1.1.4 Generation of Discrete Time Signals

A discrete time signal can be generated by the following three methods.

The methods 1 and 2 are independent of any time frame but method 3 depends critically on time.

1. Generate a set of numbers and arrange them as a sequence.

Example:

The numbers 0, 2, 4, ..., 2N form a sequence of even numbers and can be expressed as,

$$x(n) = 2n \; ; \quad 0 \le n \le N$$

2. Evaluation of a numerical recursion relation will generate a discrete signal.

Example:

$x(n) = 0.2\,x(n-1)$ with initial condition $x(0) = 1$, gives the sequence, $x(n) = 0.2^n \; ; \; 0 \le n < \infty$

When $n = 0$; $x(0) = 1$ (initial condition) $= 0.2^0$

When $n = 1$; $x(1) = 0.2\,x(1-1) = 0.2 \times x(0) = 0.2 \times 1 = 0.2^1$

When $n = 2$; $x(2) = 0.2\,x(2-1) = 0.2 \times x(1) = 0.2 \times 0.2 = 0.2^2$

When $n = 3$; $x(3) = 0.2\,x(3-1) = 0.2 \times x(2) = 0.2 \times 0.2^2 = 0.2^3$ and so on

$$\therefore \; x(n) = 0.2^n \; ; \; 0 \le n < \infty$$

3. A third method is by uniformly sampling a continuous time signal and using the amplitudes of the samples to form a sequence.

Let, x(t) = Continuous time signal

The discrete time signal can be obtained by replacing t by nT.

$$\therefore \text{ Discrete signal, } x(nT) = x(t)\big|_{t=nT} \quad ; \quad -\infty < n < \infty$$

where, T is the sampling interval.

The generation of discrete signal by sampling an analog signal is shown in Fig 1.2.

1.1.5 Digital Signal

Digital signal: *It is same as a discrete signal except that the magnitude of the signal is quantized. The magnitude of the signal can take one of the values in a set of quantized values. Here quantization is necessary to represent the signal in binary codes.*

The generation of a discrete time signal by sampling a continuous time signal and then quantizing the samples in order to convert the signal to digital signal is shown in the following example.

Let, $x(t)$ = Continuous time signal

T = Sampling time

A typical continuous time signal and the sampling of this continuous time signal at uniform interval are shown in Figs 1.2a and 1.2b, respectively. The samples of the continuous time signal as a function of sampling time instants are shown in Fig 1.2c. Here, 1T, 2T, 3T, ... etc., represent sampling time instants and the value of the samples are functions of these sampling time instants.

When $t = 0$; $x(t) = 0$

When $t = 1T$; $x(t) = 0.1$

When $t = 2T$; $x(t) = 0.3$

When $t = 3T$; $x(t) = 0.35$

When $t = 4T$; $x(t) = 0.55$

When $t = 5T$; $x(t) = 0.8$

When $t = 6T$; $x(t) = 0.8$

When $t = 7T$; $x(t) = 0.9$

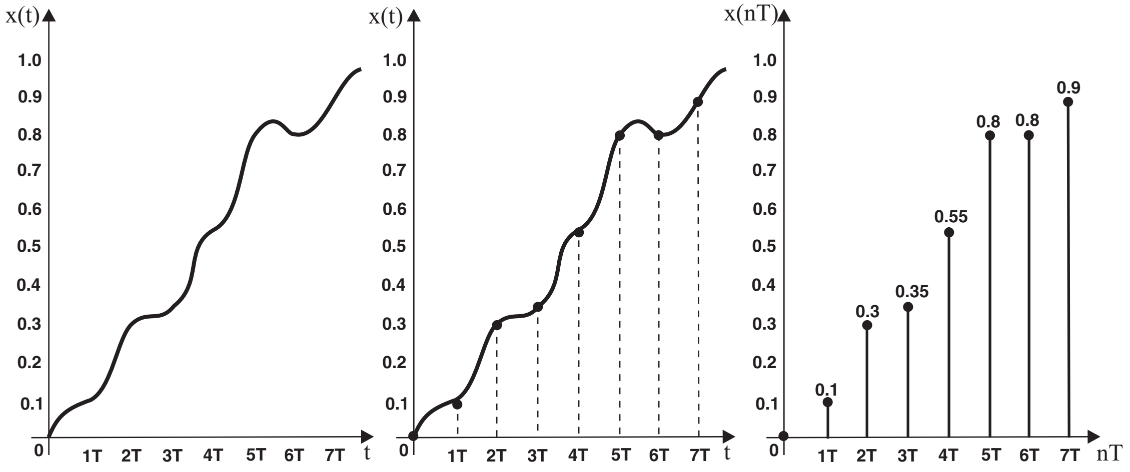


Fig a. *Fig b.* *Fig c.*

Fig 1.2: *Sampling a continuous time signal to generate discrete time signal.*

In general, the sampling time instants can be represented as, "nT", where "n" is an integer. When we drop the sampling time "T", then the samples are functions of the integer variable "n" alone. Therefore, the samples of the continuous time signal will be a discrete time signal, denoted as $x(n)$, which is a function of an integer variable "n" as shown ahead:

Here the discrete signal x(n) is defined for, $n = 0, 1, 2, 3, 4, 5, 6, 7$

$$\therefore\quad x(0) = 0 \qquad x(4) = 0.55$$
$$x(1) = 0.1 \qquad x(5) = 0.8$$
$$x(2) = 0.3 \qquad x(6) = 0.8$$
$$x(3) = 0.35 \qquad x(7) = 0.9$$

$$\therefore\quad x(n) = \{\ 0,\ 0.1,\ 0.3,\ 0.35,\ 0.55,\ 0.8,\ 0.8,\ 0.9\ \}$$

The values of the samples lies in the range of 0 to 1.

Let us choose 3-bit binary to represent the values of the samples in binary code. Now, the possible binary codes are $2^3 = 8$, and so the range can be divided into eight quantization levels, and each sample is assigned one quantization level as shown in the Table 1.2.

Table 1.2: Quantization of Analog value 0 to 1 using 3 Bit Binary

Quantization Level (R = Range = 1)	**Binary Code**	**Range Represented by Quantization Level for Quantization by Truncation**
$0 \times \frac{R}{2^3} = 0 \times \frac{1}{8} = 0$	000	$0.000 \le x(n) < 0.125 \Rightarrow 0.000$
$1 \times \frac{R}{2^3} = 1 \times \frac{1}{8} = 0.125$	001	$0.125 \le x(n) < 0.250 \Rightarrow 0.125$
$2 \times \frac{R}{2^3} = 2 \times \frac{1}{8} = 0.25$	010	$0.250 \le x(n) < 0.375 \Rightarrow 0.250$
$3 \times \frac{R}{2^3} = 3 \times \frac{1}{8} = 0.375$	011	$0.375 \le x(n) < 0.500 \Rightarrow 0.375$
$4 \times \frac{R}{2^3} = 4 \times \frac{1}{8} = 0.5$	100	$0.500 \le x(n) < 0.625 \Rightarrow 0.500$
$5 \times \frac{R}{2^3} = 5 \times \frac{1}{8} = 0.625$	101	$0.625 \le x(n) < 0.750 \Rightarrow 0.625$
$6 \times \frac{R}{2^3} = 6 \times \frac{1}{8} = 0.75$	110	$0.750 \le x(n) < 0.875 \Rightarrow 0.750$
$7 \times \frac{R}{2^3} = 7 \times \frac{1}{8} = 0.875$	111	$0.875 \le x(n) < 0.100 \Rightarrow 0.875$

Let, $x_q(n)$ = Quantized discrete time signal.

$x_c(n)$ = Quantized and coded discrete time signal.

$$x(0) = 0 \xrightarrow{\text{Quantization}} x_q(0) = 0 \xrightarrow{\text{Coding}} x_c(0) = 0\ 0\ 0$$
$$x(1) = 0.1 \xrightarrow{\text{Quantization}} x_q(1) = 0 \xrightarrow{\text{Coding}} x_c(1) = 0\ 0\ 0$$
$$x(2) = 0.3 \xrightarrow{\text{Quantization}} x_q(2) = 0.25 \xrightarrow{\text{Coding}} x_c(2) = 0\ 1\ 0$$
$$x(3) = 0.35 \xrightarrow{\text{Quantization}} x_q(3) = 0.25 \xrightarrow{\text{Coding}} x_c(3) = 0\ 1\ 0$$
$$x(4) = 0.55 \xrightarrow{\text{Quantization}} x_q(4) = 0.5 \xrightarrow{\text{Coding}} x_c(4) = 1\ 0\ 0$$
$$x(5) = 0.8 \xrightarrow{\text{Quantization}} x_q(5) = 0.75 \xrightarrow{\text{Coding}} x_c(5) = 1\ 1\ 0$$

$$x(6) = 0.8 \xrightarrow{\text{Quantization}} x_q(6) = 0.75 \xrightarrow{\text{Coding}} x_c(6) = 1\,1\,0$$

$$x(7) = 0.9 \xrightarrow{\text{Quantization}} x_q(7) = 0.875 \xrightarrow{\text{Coding}} x_c(7) = 1\,1\,1$$

$$\therefore \quad x_q(n) = \{\, 0,\ 0,\ 0.25,\ 0.25,\ 0.5,\ 0.75,\ 0.75,\ 0.875 \,\}$$

$$x_c(n) = \{\, 000,\ 000,\ 010,\ 010,\ 100,\ 110,\ 110,\ 111 \}$$

The quantized and coded discrete time signal $x_c(n)$ is called digital signal.

1.1.6 Mathematical Representation of Discrete Time Signals

The discrete time signal can be represented by the following methods.

1. Functional Representation

In functional representation, the signal is represented as a mathematical equation, as shown in the following example.

$$\begin{aligned} x(n) &= -0.5 \ ; \ n = -2 \\ &= 1.0 \ ; \ n = -1 \\ &= -1.0 \ ; \ n = 0 \\ &= 0.6 \ ; \ n = 1 \\ &= 1.2 \ ; \ n = 2 \\ &= 1.5 \ ; \ n = 3 \\ &= 0 \ ; \ \text{other } n \end{aligned}$$

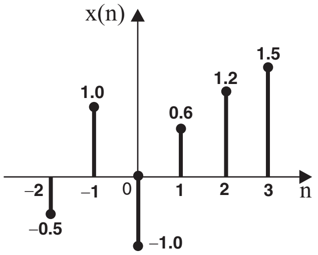


Fig 1.3: *Graphical representation of a discrete time signal.*

2. Graphical Representation

In graphical representation, the signal is represented in a two-dimensional plane. The independent variable is represented in the horizontal axis and the value of the signal is represented in the vertical axis as shown in Fig 1.3.

3. Tabular Representation

In tabular representation, two rows of a table are used to represent a discrete time signal. In the first row, the independent variable "n" is tabulated and in the second row the value of the signal for each value of "n" are tabulated as shown in the following table.

n	....	–2	–1	0	1	2	3	....
x(n)	....	–0.5	1.0	–1.0	0.6	1.2	1.5	....

4. Sequence Representation

In sequence representation, the discrete time signal is represented as a one-dimensional array as shown in the following examples.

An infinite duration discrete time signal with time origin, n = 0, indicated by symbol ↑ is represented as,

$$x(n) = \{ \ldots.. -0.5,\ \underset{\uparrow}{1.0},\ -1.0,\ 0.6,\ 1.2,\ 1.5,\ \ldots.. \}$$

An infinite duration discrete time signal that satisfies the condition x(n) = 0 for n < 0 is represented as,

$$x(n) = \{ \underset{\uparrow}{-1.0},\ 0.6,\ 1.2,\ 1.5,\ \ldots \} \qquad \text{or} \qquad x(n) = \{-1.0,\ 0.6,\ 1.2,\ 1.5,\ \ldots \}$$

An infinite duration discrete time signal that satisfies the condition x(n) = 0 for n > 0 is represented as,

$$x(n) = \{ \ldots - 0.5,\ 1.0,\ \underset{\uparrow}{-1.0} \}$$

A finite duration discrete time signal with time origin, n = 0, indicated by symbol ↑ is represented as,

$$x(n) = \{ -0.5,\ 1.0,\ \underset{\uparrow}{-1.0},\ 0.6,\ 1.2,\ 1.5 \}$$

A finite duration discrete time signal that satisfies the condition x(n) = 0 for n < 0 is represented as,

$$x(n) = \{ \underset{\uparrow}{-1.0},\ -0.6,\ 1.2,\ 1.5 \} \qquad \text{or} \qquad x(n) = \{ -1.0,\ 0.6,\ 1.2,\ 1.5 \}$$

A finite duration discrete time signal that satisfies the condition x(n) = 0 for n > 0 is represented as,

$$x(n) = \{ -0.5,\ 1.0,\ \underset{\uparrow}{-1.0} \}$$

1.1.7 Standard Discrete Time Signals

1. Discrete Impulse Signal (Unit Impulse Sequence)

$$\begin{aligned} \text{Impulse signal, } \delta(n) &= 1 \ ; \ n = 0 \\ &= 0 \ ; \ n \neq 0 \end{aligned}$$

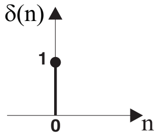


Fig 1.4: *Discrete impulse signal.*

2. Discrete Unit Step Signal (Unit Step Sequence)

$$\begin{aligned} \text{Unit step signal, } u(n) &= 1 \ ; \ n \geq 0 \\ &= 0 \ ; \ n < 0 \end{aligned}$$

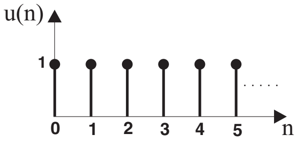


Fig 1.5: *Unit step signal.*

3. Discrete Ramp Signal (Ramp Sequence)

$$\begin{aligned} \text{Ramp signal, } U_r(n) &= n \ ; \ n \geq 0 \\ &= 0 \ ; \ n < 0 \end{aligned}$$

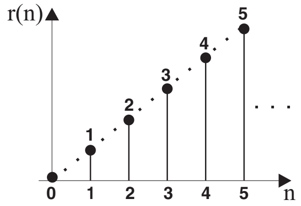


Fig 1.6: *Ramp signal.*

4. Discrete Exponential Signal (Exponential Sequence)

$$\begin{aligned} \text{Exponential signal, } g(n) &= a^n \ ; \ n \geq 0 \\ &= 0 \ \ ; \ n < 0 \end{aligned}$$

Fig a: *Decreasing exponential signal.*

Fig b: *Increasing exponential signal.*

Fig 1.7: *Exponential signal.*

5. Discrete Signum Signal (Signum Sequence)

$$\begin{aligned} \mathrm{sgn}(n) &= 1 \quad ; \; n > 0 \\ &= 0 \quad ; \; n = 0 \\ &= -1 \quad ; \; n < 0 \end{aligned}$$

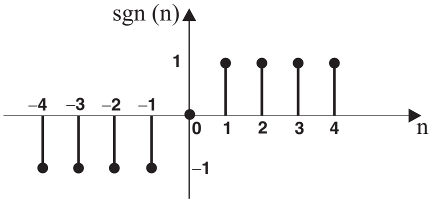


Fig 1.8: *Discrete signum signal.*

6. Discrete Rectangular Signal (Rectangular Sequence)

$$\begin{aligned} \mathrm{rect}\,(n) &= 1 \quad ; \; -N < n < N \\ &= 0 \quad ; \; n < -N \text{ and } n > N \end{aligned}$$

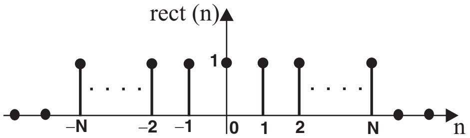


Fig 1.9: *Discrete rectangular signal.*

7. Discrete Sinusoidal Signal (Sinusoidal Sequence)

The discrete time sinusoidal signal may be expressed as,

$$x(n) = A\cos(\omega_0 n + \theta) \; ; \text{ for n in the range } -\infty < n < +\infty$$

$$x(n) = A\sin(\omega_0 n + \theta) \; ; \text{ for n in the range } -\infty < n < +\infty$$

where, ω_0 = Frequency in *radians/sample* ; θ = Phase in *radians*

$f_0 = \frac{\omega_0}{2\pi}$ = Frequency in *cycles/sample*

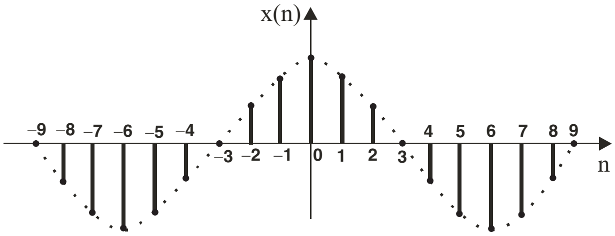


Fig a: *Discrete time sinusoidal(cosinusoidal) signal representedby equation $x(n) = A\cos(\omega_0 n)$.*

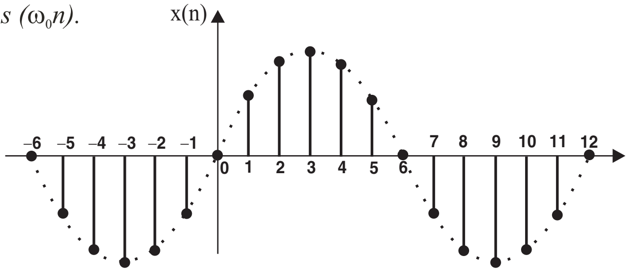


Fig b: *Discrete time sinusoidal signal represented by equation $x(n) = A\sin(\omega_0 n)$.*

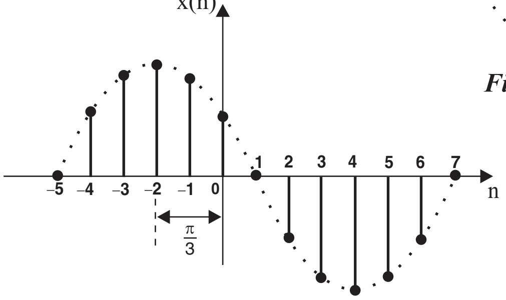


Fig c: *Discrete time sinusoidal signal represented by equation,*

$$x(n) = A\cos\left(\frac{\pi}{6}n + \frac{\pi}{3}\right) \; ; \omega_0 = \frac{\pi}{6} \; , \theta = \frac{\pi}{3}$$

Fig 1.10: *Discrete time sinusoidal signals.*

Properties of Discrete Time Sinusoid

1. A discrete time sinusoid is periodic only if its frequency f is a rational number, (i.e., ratio of two integers).
2. Discrete time sinusoids whose frequencies are separated by integer multiples of 2π are identical.

$\therefore$ $x(n) = A\cos[(\omega_0 + 2\pi k)\,n + \theta]$, for $k = 0, 1, 2, \ldots\ldots$ are identical in the interval $-\pi \le \omega_0 \le \pi$ and so they are indistinguishable.

Proof:

$$\cos[(\omega_0 + 2\pi k)\,n + \theta] = \cos(\omega_0 n + 2\pi nk + \theta) = \cos[(\omega_0 n + \theta) + 2\pi nk]$$

$$= \cos(\omega_0 n + \theta)\cos 2\pi nk - \sin(\omega_0 n + \theta)\sin 2\pi nk$$

Since n and k are integers, $\cos 2\pi nk = 1$ and $\sin 2\pi nk = 0$

Cos (A + B) = cosA cosB – sinA sinB

$\therefore \cos[(\omega_0 + 2\pi k)\,n + \theta] = \cos(\omega_0 n + \theta)$, for k = 0, 1, 2, 3, ...

Conclusion

i) The sequences of any two sinusoids with frequencies in the range, $-\pi \le \omega_o \le \pi$ (or $-1/2 \le f_0 \le 1/2$), are distinct.

$$[-\pi \le \omega \le \pi \xrightarrow{\text{divide by } 2\pi} -1/2 \le f \le 1/2]$$

ii) Any discrete time sinusoid with frequency $\omega_o > |\pi|$ (or $f_0 > |1/2|$) will be identical to another discrete time sinusoid with frequency $\omega_o < |\pi|$ (or $f_0 < |1/2|$).

8. Discrete Time Complex Exponential Signal

The discrete time complex exponential signal is defined as,

$$x(n) = a^n e^{j(\omega_0 n + \theta)} = a^n[\cos(\omega_0 n + \theta) + j\sin(\omega_0 n + \theta)]$$

$$= a^n\cos(\omega_0 n + \theta) + j\,a^n\sin(\omega_0 n + \theta) = x_r(n) + j\,x_i(n)$$

where, $x_r(n)$ = Real part of $x(n) = a^n\cos(\omega_0 n + \theta)$

$x_i(n)$ = Imaginary part of $x(n) = a^n\sin(\omega_0 n + \theta)$

The real part of x(n) will give an exponentially increasing cosinusoid sequence for a > 1 and exponentially decreasing cosinusoid sequence for 0 < a < 1.

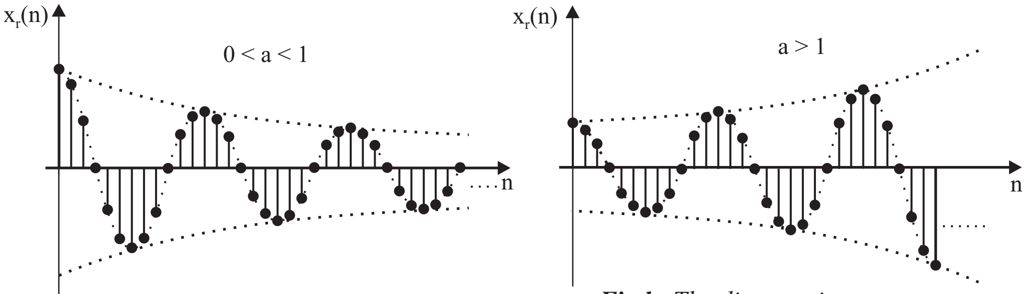


Fig a: *The discrete time sequence represented by the equation, $x_r(n) = a^n\cos\omega_0 n$ for $0 < a < 1$.*

Fig b: *The discrete time sequence represented by the equation, $x_r(n) = a^n\cos\omega_0 n$ for $a > 1$.*

Fig 1.11: *Real part of complex exponential signal.*

The imaginary part of x(n) will give rise to an exponentially increasing sinusoid sequence for a > 1 and exponentially decreasing sinusoid sequence for 0 < a < 1.

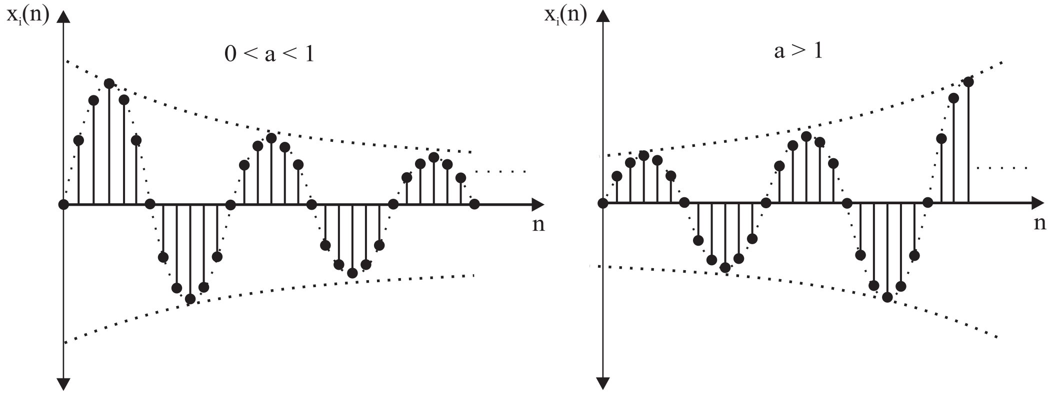


Fig a: *The discrete time sequence represented by the equation,* $x_i(n) = a^n \sin\omega_0 n$ *for* $0 < a < 1$.

Fig b: *The discrete time sequence represented by the equation,* $x_i(n) = a^n \sin\omega_0 n$ *for* $a > 1$.

Fig 1.12: *Imaginary part of complex exponential signal.*

1.1.8 Systems

System: *Any process that exhibits a cause and effect relation can be called a system.*

A system will have an input signal and an output signal. The output signal will be a processed version of the input signal. A system is interconnection of either hardware devices or software/algorithm. A system is denoted by the letter '$\mathcal{H}$'. The diagrammatic representation of a system is shown in Fig 1.13.

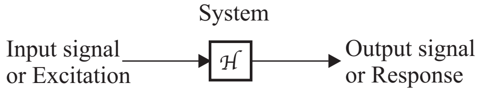


Fig 1.13: *Representation of a system.*

The operation performed by a system on the input signal to produce the output signal can be expressed as,

$$\text{Output} = \mathcal{H}\{\text{Input}\}$$

where $\mathcal{H}$ denotes the system operation (also called ***system operator***).

The systems can be classified in many ways.

Depending on the type of energy used to operate the systems, the systems can be classified into electrical systems, mechanical systems, thermal systems, hydraulic systems, etc.

Depending on the type of input and output signals, the systems can be classified into Continuous time systems and Discrete time systems.

1.1.9 Continuous Time System

Continuous time system: *A system which process and produce continuous time signal is called continuous time system.*

The input and output signals of a continuous time system are continuous time signals. The diagrammatic representation of a continuous time system is shown in Fig 1.14.

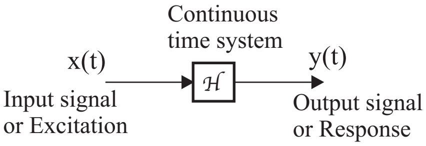


Fig 1.14: Representation of continuous time system.

where, $\mathcal{H}$ = System operator

x(t) = Continuous time input signal

y(t) = Continuous time output signal

The operation performed by a continuous time system on input x(t) to produce output or response y(t) can be expressed as,

Response, $y(t) = \mathcal{H}\{x(t)\}$ .....(1.1)

Linear Time Invariant Continuous Time (LTI-CT) System: *A continuous time system which satisfies the properties of linearity and time invariance is called a Linear Time Invariant Continuous Time (LTI-CT) system.*

Most of the practical systems that we encounter in science and engineering are LTI systems.

Mathematical representation of LTI-CT system

The input-output relation of an LTI continuous time system is represented by constant coefficient differential equation shown below:

$$a_0 \frac{d^N}{dt^N} y(t) + a_1 \frac{d^{N-1}}{dt^{N-1}} y(t) + a_2 \frac{d^{N-2}}{dt^{N-2}} y(t) + ... + a_{N-1} \frac{d}{dt} y(t) + a_N y(t) = b_0 \frac{d^M}{dt^M} x(t)$$

$$+ b_1 \frac{d^{M-1}}{dt^{M-1}} x(t) + b_2 \frac{d^{M-2}}{dt^{M-2}} x(t) + ... + b_{M-1} \frac{d}{dt} x(t) + b_M x(t) \quad .....(1.2)$$

where, N = Order of the system, $M \leq N$, $a_0 = 1$.

a_N, b_0, b_1, ... b_m are constant coefficient.

The solution of the above differential equation is the response y(t) of the continuous time system, for the input x(t).

1.1.10 Discrete Time System

Discrete time system: *A system which processes an input discrete time signal and produce an output discrete time signal is called discrete time system.*

The input and output signals of a discrete time system are discrete time signals. The diagrammatic representation of a discrete time system is shown in Fig 1.15.

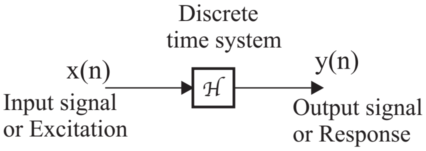


Fig 1.15: Representation of a discrete time system.

where, $\mathcal{H}$ = System operator

x(n) = Discrete time input signal

y(n) = Discrete time output signal

The operation performed by a discrete time system on input x(n) to produce output or response y(n) can be expressed as,

Response, $y(n) = \mathcal{H}\{x(n)\}$ .....(1.3)

Linear Time Invariant Discrete Time (LTI-DT) System: *A discrete time system which satisfies the properties of linearity and time invariance is called a Linear Time Invariant Discrete Time (LTI-DT) system.*

Impulse and Step Response

Impulse response: *When the input to a discrete time system is a unit impulse δ(n), then the output is called an impulse response of the system and is denoted by h(n).*

$\therefore$ Impulse Response, $h(n) = \mathcal{H}\{\delta(n)\}$ .....(1.4)

Fig a: *Discrete time system with impulse input.*

Fig b: *Discrete time system with unit step input.*

Fig 1.16: *Discrete time systems with impulse and step input.*

Step response: *When the input to a discrete time system is a unit step signal u(n), then the output is called step response of the system and is denoted by s(n).*

$\therefore$ Step Response, $s(n) = \mathcal{H}\{u(n)\}$ .....(1.5)

1.1.11 Difference Equation Governing Discrete Time System (Mathematical Representation of Discrete Time System)

The input-output relation of an LTI discrete time system is represented by the constant coefficient difference equation shown below:

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m) \quad .....(1.6)$$

where, N = Order of the system, $M \le N$.

a_m, b_m are constant coefficients.

The solution of the above difference equation is the response y(n) of the discrete time system, for the input x(n).

The mathematical equation governing the discrete time system can be developed as shown below:

The response of a discrete time system at any time instant depends on the present and past inputs and past outputs.

Let us consider the response at n = 0. Let us assume a relaxed system and so at n = 0, there is no past input or output. Therefore, the response at n = 0, is a function of the present input alone.

$\therefore$ $y(0) = F[x(0)]$

Let us consider the response at n =1. Now the present input is x(1), the past input is x(0) and the past output is y(0). Therefore, the response at n = 1, is a function of x(1), x(0), y(0).

$\therefore$ $y(1) = F[y(0), x(1), x(0)]$

Let us consider the response at n = 2. Now the present input is x(2), the past inputs are x(1) and x(0) and the past outputs are y(1) and y(0). Therefore, the response at n = 2, is a function of x(2), x(1), x(0), y(1), y(0).

$$\therefore\ y(2) = F[y(1), y(0), x(2), x(1), x(0)]$$

Similarly, at n = 3, $y(3) = F[y(2), y(1), y(0), x(3), x(2), x(1), x(0)]$

at n = 4, $y(4) = F[y(3), y(2), y(1), y(0), x(4), x(3), x(2), x(1), x(0)]$ and so on.

In general, at any time instant n,

$$y(n) = F[y(n-1), y(n-2), y(n-3), ...y(1), y(0), x(n), x(n-1),$$
$$x(n-2), x(n-3) ..... x(1), x(0)]$$

For an LTI system, the response y(n) can be expressed as a weighted summation of dependent terms. Therefore, the above equation can be written as,

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) - ....$$
$$+ b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + b_3\, x(n-3) + ... \quad .....(1.7)$$

where, $a_1, a_2, a_3, ...$ and $b_0, b_1, b_2, b_3, ...$ are constants.

> ***Note:*** *Negative constants are inserted for output signals, because output signals are feed back from the output to input. Positive constants are inserted for input signals, because input signals are fed forward from the input to output.*

Practically, the response y(n) at any time instant n, may depend on N number of past outputs, present input and M number of past inputs where $M \leq N$. Hence, equation (1.7) can be written as,

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) - .... - a_N\, y(n-N)$$
$$+ b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + b_3\, x(n-3) + ... + b_M\, x(n-M)$$

$$\therefore\quad y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m) \quad .....(1.8)$$

The equation (1.8) is a constant coefficient ***difference equation*** governing the input-output relation of an LTI discrete time system.

In equation (1.8), the value of "N" gives the ***order*** of the system.

If N = 1, the discrete time system is called 1st order system

If N = 2, the discrete time system is called 2nd order system

If N = 3, the discrete time system is called 3rd order system and so on.

The general difference equation governing 1st order discrete time LTI system is,

$$y(n) = -a_1\, y(n-1) + b_0\, x(n) + b_1\, x(n-1)$$

The general difference equation governing 2nd order discrete time LTI system is,

$$y(n) = -a_2\, y(n-2) - a_1\, y(n-1) + b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2)$$

The general difference equation governing 3nd order discrete time LTI system is,

$$y(n) = -a_3\, y(n-3) - a_2\, y(n-2) - a_1\, y(n-1) + b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + b_3\, x(n-3)$$

1.1.12 Block Diagram and Signal Flow Graph Representation of Discrete Time System

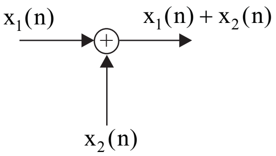
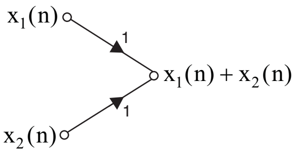
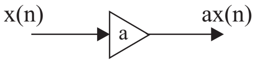
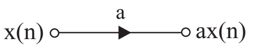
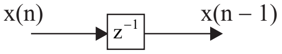
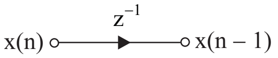
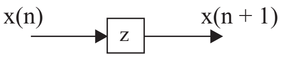
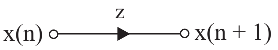
The discrete time system can be represented diagrammatically by a ***block diagram or Signal flow graph***. These diagrammatic representations are useful for physical implementation of a discrete time systems in hardware or software.

The basic elements employed in a block diagram are adder, constant multiplier, unit delay element and unit advance element.

Adder	**:** An adder is used to represent addition of two discrete time signals.
Constant multiplier	**:** A constant multiplier is used to represent multiplication of a scaling factor (constant) to a discrete time signal.
Unit delay element	**:** A unit delay element is used to represent the delay of samples of a discrete time signal by one sampling time.
Unit advance element	**:** A unit advance element is used to represent the advance of samples of a discrete time signal by one sampling time.

The symbolic representation of the basic elements of a block diagram and signal flow graph are listed in Table 1.3.

Table 1.3: Basic Elements of Block Diagram

Element	Block Diagram Representation	Signal Flow Graph Representation
Adder		
Constant multiplier		
Unit delay element		
Unit advance element		

Example 1.1

Construct the block diagram and signal flow graph of discrete time systems whose input-output relations are described by the following difference equations.

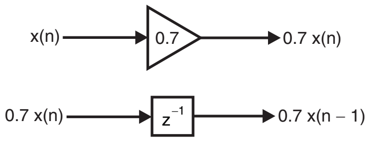
a) $y(n) = 0.7\,x(n) + 0.7\,x(n-1)$ **b)** $y(n) = 0.4\,y(n-1) + x(n) - 3\,x(n-2)$

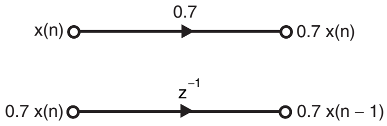
c) $y(n) = 0.2\,y(n-1) + 0.7\,x(n) + 0.9\,x(n-1)$

Solution

a) Given that, $y(n) = 0.7\,x(n) + 0.7\,x(n-1)$

The individual terms of the given equation are 0.7 x(n) and 0.7 x(n – 1). They are represented by basic elements as shown ahead.





The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

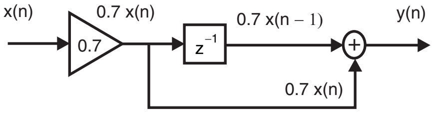


***Fig 1:** Block diagram of the system described by the equation y(n) = 0.7 x(n) + 0.7 x(n – 1).*

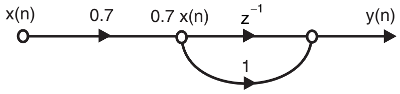
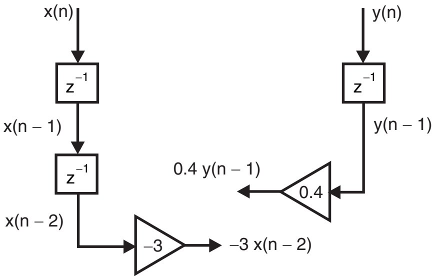


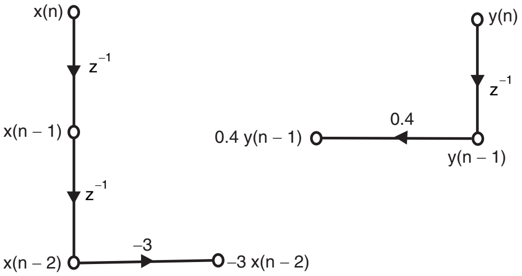
***Fig 2:** Signal flow graph of the system described by the equation y(n) = 0.7 x(n) + 0.7 x(n – 1).*

---

b) Given that, y(n) = 0.4 y(n – 1) + x(n) – 3 x(n – 2)

The individual terms of the given equation are 0.4 y(n – 1) and – 3 x(n – 2). They are represented by basic elements as shown below.





The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

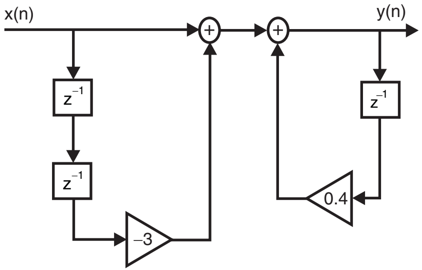


***Fig 3:** Block diagram of the system described by the equation y(n) = 0.4 y(n – 1) + x(n) – 3 x(n – 2).*

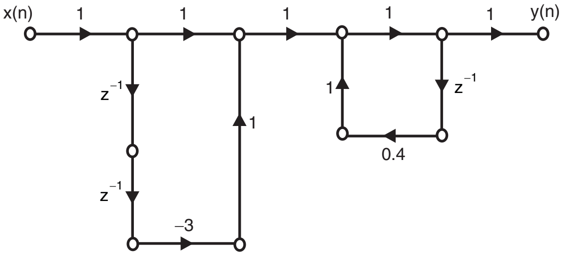
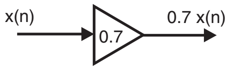


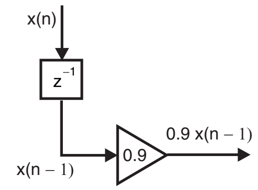
***Fig 4:** Signal flow graph of the system described by the equation y(n) = 0.4 y(n – 1) + x(n) – 3 x(n – 2).*

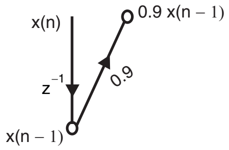
---

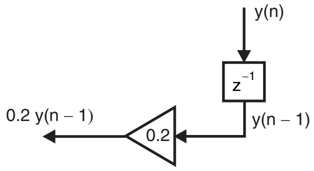
c) Given that, y(n) = 0.2 y(n – 1) + 0.7 x(n) + 0.9 x(n – 1)

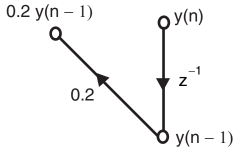
The individual terms of the given equation are 0.2 y(n – 1), 0.7 x(n) and 0.9 x(n – 1). They are represented by basic elements as shown below.











The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

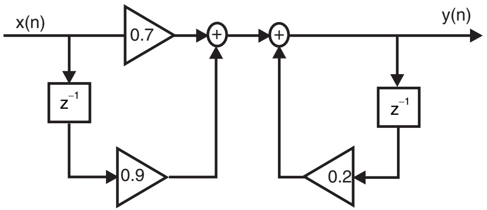


Fig 5: *Block diagram of the system described by the equation y(n) = 0.2 y(n – 1) + 0.7 x(n)+ 0.9 x(n – 1).*

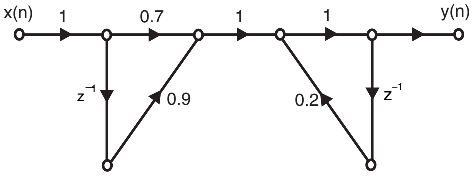


Fig 6: *Signal flow graph of the system described by the equation y(n) = 0.2 y(n – 1) + 0.7 x(n) + 0.9 x(n – 1).*

1.1.13 Convolution

Discrete or Linear Convolution

The ***discrete*** or ***linear convolution*** of two discrete time sequences $x_1(n)$ and $x_2(n)$ is defined as,

$$\boxed{x_3(n) = \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m) \qquad \text{or} \qquad x_3(n) = \sum_{m=-\infty}^{+\infty} x_2(m)\, x_1(n-m)} \qquad .....(1.9)$$

where, $x_3(n)$ is the sequence obtained by convolving $x_1(n)$ and $x_2(n)$

m is a dummy variable.

The convolution relation of equation (1.9) can be symbolically expressed as,

$$x_3(n) = x_1(n) * x_2(n) = x_2(n) * x_1(n) \qquad .....(1.10)$$

where, the symbol $*$ indicates the convolution operation.

In linear convolution, the sequences $x_1(n)$ and $x_2(n)$ are nonperiodic sequences and the sequence $x_3(n)$ obtained by convolution is also nonperiodic. Hence, this convolution is also called ***aperiodic convolution.***

Procedure for Evaluating Linear Convolution

Let $x_1(n)$ and $x_2(n)$ be two discrete time sequences.

Let $x_3(n)$ be the sequence obtained by the convolution of $x_1(n)$ and $x_2(n)$.

$$\therefore\ x_3(n) = x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\ ;\ -\infty < n < +\infty \qquad .....(1.11)$$

Now each sample of $x_3(n)$ can be computed using the above equation.

The value of $x_3(n)$ at $n = q$ is obtained by replacing n by q, in the above equation.

$$\therefore\ x_3(q) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(q-m) \qquad .....(1.12)$$

The evaluation of equation (1.12) to determine the value of $x_3(n)$ at $n = q$, involves the following five steps.

1. Change of index : Change the index n in the sequences $x_1(n)$ and $x_2(n)$ to get the sequences $x_1(m)$ and $x_2(m)$.

2. Folding ***:*** Fold $x_2(m)$ about $m = 0$ to obtain $x_2(-m)$.

3. Shifting ***:*** Shift $x_2(-m)$ by q to the right if q is positive or shift $x_2(-m)$ by q to the left if q is negative to obtain $x_2(q-m)$.

4. Multiplication ***:*** Multiply $x_1(m)$ by $x_2(q-m)$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x_1(m) \times x_2(q-m)$.

5. Summation ***:*** Sum all the values of the product sequence $v_q(m)$ to obtain the value of $x_3(n)$ at $n = q$. [i.e., $x_3(q)$].

The above procedure will give the value of $x_3(n)$ at a single time instant say $n = q$. In general, we are interested in evaluating the values of the sequence $x_3(n)$ over all the time instants in the range $-\infty < n < \infty$. Hence, the steps 3, 4 and 5 given above must be repeated for all possible time shifts in the range $-\infty < n < \infty$.

Convolution of Finite Duration Sequences

In convolution of finite duration sequences it is possible to predict the length of the resultant sequence.

If the sequence $x_1(n)$ has N_1 samples and sequence $x_2(n)$ has N_2 samples then the output sequence $x_3(n)$ will be a finite duration sequence consisting of "$N_1 + N_2 - 1$" samples.

Let, length of $x_1(n) = N_1$

length of $x_2(n) = N_2$

Now, length of $x_3(n) = N_1 + N_2 - 1$

In the convolution of finite duration sequences it is possible to predict the start and end of the resultant sequence. If $x_1(n)$ starts at $n = n_1$ and $x_2(n)$ starts at $n = n_2$ then the initial value of n for $x_3(n)$ is "$n = n_1 + n_2$". The value of $x_1(n)$ for $n < n_1$ and the value of $x_2(n)$ for $n < n_2$ are then assumed to be zero. The final value of n for $x_3(n)$ is "$n = (n_1 + n_2) + (N_1 + N_2 - 2)$".

Let, $x_1(n)$ start at $n = n_1$

$x_2(n)$ start at $n = n_2$

Now, $x_3(n)$ start at $n = n_1 + n_2$

and $x_3(n)$ end at $n = (n_1 + n_2) + (N_1 + N_2 - 1) - 1$

$= (n_1 + n_2) + (N_1 + N_2 - 2)$

Linear Convolution by Tabular Method

In the tabular method, every input sequence and folded and shifted sequence is represented by a row in a table.

Let $x_1(n)$ and $x_2(n)$ be the input sequences and $x_3(n)$ be the output sequence.

1. Change the index "n" of input sequences to "m" to get $x_1(m)$ and $x_2(m)$.
2. Represent the input sequences $x_1(m)$ and $x_2(m)$ as two rows in tabular array.
3. Let us fold $x_2(m)$ to get $x_2(-m)$. Represent the folded sequence $x_2(-m)$ in the third rows of tabular array.
4. Shift the folded sequence $x_2(-m)$ to the left and represent in next row so that the product of $x_1(m)$ and shifted $x_2(-m)$ gives only one nonzero sample. Now multiply $x_1(m)$ and shifted $x_2(-m)$ to get a product sequence, and then sum up the samples of product sequence, which is the first sample of output sequence.
5. To get the next sample of output sequence, shift $x_2(-m)$ of the previous step to one position right and multiply the shifted sequence with $x_1(m)$ to get a product sequence. Now the sum of the samples of the product sequence gives the second sample of the output sequence.
6. To get subsequent samples of the output sequence, the step 5 is repeated until we get a nonzero product sequence.

Example 1.2

Determine the response of the LTI system whose input x(n) and impulse response h(n) are given by,

$x(n) = \{\underset{\uparrow}{1}, 2, 0.5, 1\}$ and $h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$

Solution

The response y(n) of the system is given by convolution of x(n) and h(n).

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m)$$

In this example the convolution operation is performed by three methods.

The input sequence starts at n = 0 and the impulse response sequence starts at n = –1. Therefore, the output sequence starts at n = 0 + (–1) = –1.

The input and impulse responses consist of 4 samples, so the output consists of 4 + 4 – 1 = 7 samples.

Tabular Method

The given sequences and the shifted sequences can be represented in the tabular array as shown below.

***Note:** The unfilled boxes in the table are considered as zeros.*

m	–3	–2	–1	0	1	2	3	4	5	6
x(m)				1	2	0.5	1			
h(m)			1	2	1	–1				
h(–m)		–1	1	2	1					
$h(-1-m) = h_{-1}(m)$	–1	1	2	1						
$h(0-m) = h_0(m)$		–1	1	2	1					
$h(1-m) = h_1(m)$			–1	1	2	1				
$h(2-m) = h_2(m)$				–1	1	2	1			
$h(3-m) = h_3(m)$					–1	1	2	1		
$h(4-m) = h_4(m)$						–1	1	2	1	
$h(5-m) = h_5(m)$							–1	1	2	1

Each sample of y(n) is computed using the convolution formula,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\,h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\,h_n(m), \text{ where } h_n(m) = h(n-m)$$

To determine a sample of y(n) at n = q, multiply the sequence x(m) and $h_q(m)$ to get a product sequence [i.e., multiply the corresponding elements of the row x(m) and $h_q(m)$]. The sum of all the samples of the product sequence gives y(q).

Here the product is valid only for m = –3 to +3.

$$\text{When } n = -1 \;;\; y(-1) = \sum_{m=-3}^{3} x(m)\,h_{-1}(m)$$

$$= x(-3)\,h_{-1}(-3) + x(-2)\,h_{-1}(-2) + x(-1)\,h_{-1}(-1) + x(0)\,h_{-1}(0) + x(1)\,h_{-1}(1) + x(2)\,h_{-1}(2) + x(3)\,h_{-1}(3) = 0+0+0+1+0+0+0 = 1$$

The samples of y(n) for other values of n are calculated as shown for n = –1.

$$\text{When } n = 0 \;;\; y(0) = \sum_{m=-2}^{3} x(m)\,h_0(m) = 0+0+2+2+0+0 = 4$$

$$\text{When } n = 1 \;;\; y(1) = \sum_{m=-1}^{3} x(m)\,h_1(m) = 0+1+4+0.5+0 = 5.5$$

$$\text{When } n = 2 \;;\; y(2) = \sum_{m=0}^{3} x(m)\,h_2(m) = -1+2+1+1 = 3$$

$$\text{When } n = 3 \;;\; y(3) = \sum_{m=0}^{4} x(m)\,h_3(m) = 0-2+0.5+2+0 = 0.5$$

$$\text{When } n = 4 \;;\; y(4) = \sum_{m=0}^{5} x(m)\,h_4(m) = 0+0-0.5+1+0+0 = 0.5$$

$$\text{When } n = 5 \;;\; y(5) = \sum_{m=0}^{6} x(m)\,h_5(m) = 0+0+0-1+0+0+0 = -1$$

The output sequence , $y(n) = \{1, \underset{\uparrow}{4}, 5.5, 3, 0.5, 0.5, -1\}$

1.1.14 Circular Convolution

Circular Representation and Circular Shift of Discrete Time Signal

Consider a finite duration sequence x(n) and its periodic extension $x_p(n)$. The periodic extension of x(n) can be expressed as $x_p(n) = x(n + N)$, where N is the periodicity. Let N = 4. The sequence x(n) and its periodic extension are shown in Fig 1.17.

Let, x(n) = 1 ; n = 0
= 2 ; n = 1
= 3 ; n = 2
= 4 ; n = 3

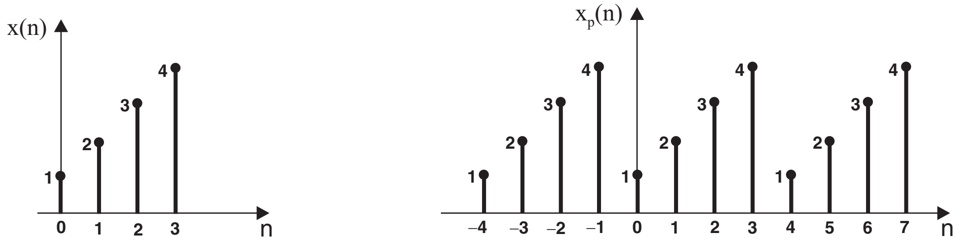


Fig a: *Finite duration sequence x(n).* ***Fig b:*** *Periodic extension of x(n).*

Fig 1.17: *A finite duration sequence and its periodic extension.*

Let us delay the periodic sequence $x_p(n)$ by two units of time as shown in Fig 1.18a. (For delay, the sequence is shifted right). Let us denote one period of this delayed sequence by $x_1(n)$. One period of the delayed sequence is shown in Fig 1.18b.

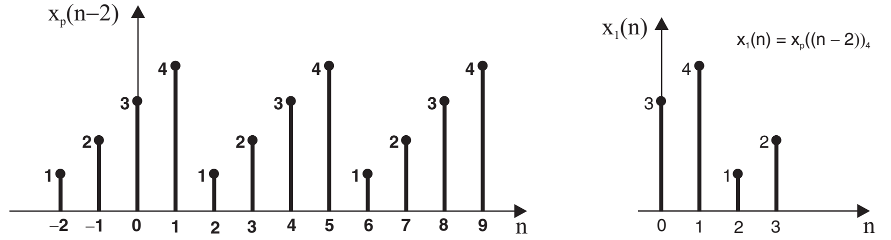


Fig a: ***x_p****(n) delayed by two units of time.* ***Fig b:*** *One period of $x_p(n-2)$.*

Fig 1.18: *Delayed version of $x_p(n)$.*

The sequence $x_1(n)$ can be represented by $x_p(n-2, (\text{mod } 4))$, or $x_p((n-2))_4$, where mod 4 indicates that the sequence repeats after 4 samples. The relation between the original sequence x(n) and one period of the delayed sequence $x_1(n)$ is shown below:

$$x_1(n) = x_p(n - 2, (\text{mod } 4)) = x_p((n-2))_4$$

$\therefore$ When $n = 0$; $x_1(0) = x_p((0-2))_4 = x_p((-2))_4 = x(2) = 3$

When $n = 1$; $x_1(1) = x_p((1-2))_4 = x_p((-1))_4 = x(3) = 4$

When $n = 2$; $x_1(2) = x_p((2-2))_4 = x_p((0))_4 = x(0) = 1$

When $n = 3$; $x_1(3) = x_p((3-2))_4 = x_p((1))_4 = x(1) = 2$

The periodic sequences $x_p(n)$ and $x_1(n)$ can be represented as points on a circle as shown in Fig 1.19. From Fig 1.19 we can say that $x_1(n)$ is simply $x_p(n)$ shifted circularly by two units in time where the counter clockwise (anticlockwise) direction has been arbitrarily selected for right shift or delay.

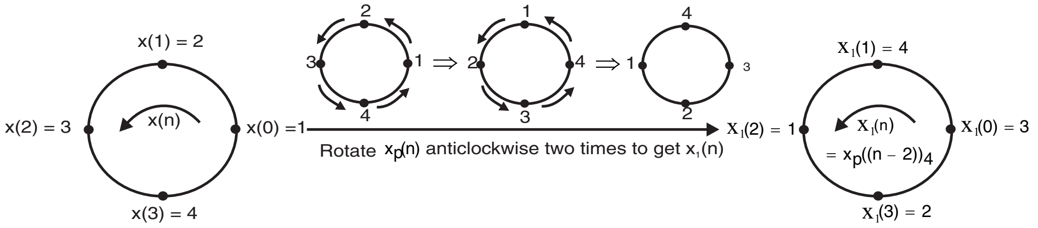


Fig a: *Circular representation of x(n).* ***Fig b:*** *Circular representation of $x_1(n)$.*

Fig 1.19: *Circular representation of a signal and its delayed version.*

Let us advance the periodic sequence $x_p(n)$ by three units of time as shown in Fig 1.20a. Let us denote one period of this advanced sequence by $x_2(n)$. One period of the advanced sequence is shown in Fig 1.20b.

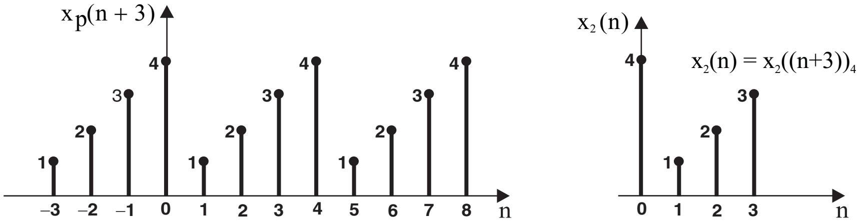


Fig a: *$x_p(n)$ advanced by three units of time.*

Fig b: *One period of $x_p(n + 3)$.*

Fig 1.20: *Advanced version of $x_p(n)$.*

The sequence $x_2(n)$ can be represented by $x_p(n+3, (\text{mod } 4))$ or $x_p((n+3))_4$, where mod 4 indicates that the sequence repeats after 4 samples. The relation between the original sequence x(n) and one period of the advanced sequence $x_2(n)$ is shown below:

$$x_2(n) = x_p(n+3, (\text{mod } 4)) = x_p((n+3))_4$$

$\therefore$ When $n = 0$; $x_2(0) = x_p((0+3))_4 = x_p((3))_4 = x(3) = 4$

When $n = 1$; $x_2(1) = x_p((1+3))_4 = x_p((4))_4 = x(0) = 1$

When $n = 2$; $x_2(2) = x_p((2+3))_4 = x_p((5))_4 = x(1) = 2$

When $n = 3$; $x_2(3) = x_p((3+3))_4 = x_p((6))_4 = x(2) = 3$

The periodic sequences $x_p(n)$ and $x_2(n)$ can be represented as points on a circle as shown in Fig 1.21. From Fig 1.21 we can say that $x_2(n)$ is simply $x_p(n)$ shifted circularly by three units in time where clockwise direction has been selected for left shift or advance.

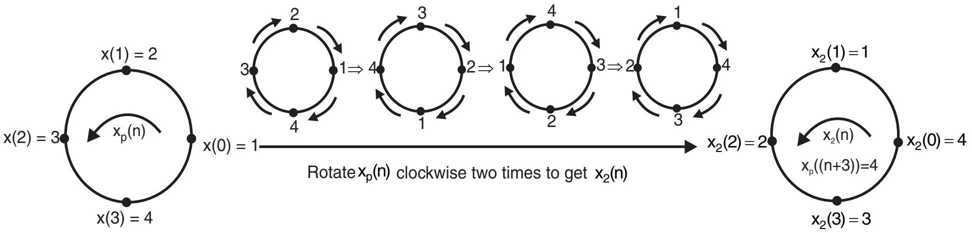


Fig a: *Circular representation of x(n).*

Fig b: *Circular representation of $x_2(n)$.*

Fig 1.21: *Circular representation of a signal and its advanced version.*

Thus, we conclude that a circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension and vice versa. If a nonperiodic N-point sequence is represented on the circumference of a circle, then it becomes a periodic sequence of periodicity N. When the sequence is shifted circularly, the samples repeat after N shifts. This is similar to modulo-N operation. Hence, in general, the circular shift may be represented by the index mod-N. Let x(n) be an N-point sequence represented on a circle and x'(n) be its ***circularly shifted sequence*** by m units of time.

$$\text{Now } x'(n) = x(n-m, \text{mod } N) \equiv x((n-m))_N \qquad .....(1.13)$$

When m is positive, equation (1.13) represents delayed sequence and when m is negative, equation (1.13) represents advanced sequence.

Circular Symmetries of Discrete Time Signal

The circular representation of a sequence and the resulting periodicity gives rise to new definitions for even symmetry, odd symmetry and time reversal of the sequence.

An N-point sequence is called even if it is symmetric about the point zero on the circle. This implies that,

$$x(N-n) = x(n) \quad ; \text{ for } \quad 0 \le n \le N-1 \qquad .....(1.14)$$

An N-point sequence is called odd if it is antisymmetric about the point zero on the circle. This implies that,

$$x(N-n) = -\, x(n) \quad ; \text{ for } \quad 0 \le n \le N-1 \qquad .....(1.15)$$

The time reversal of a N-point sequence is obtained by reversing its sample about the point zero on the circle. Thus, the sequence x(–n, (mod N)) is simply written as,

$$x\,(-n, (\text{mod } N)) = x(N-n) \quad ; \text{ for } \quad 0 \le n \le N-1 \qquad .....(1.16)$$

This time reversal is equivalent to plotting x(n) in a clockwise direction on a circle, as shown in Fig 1.22.

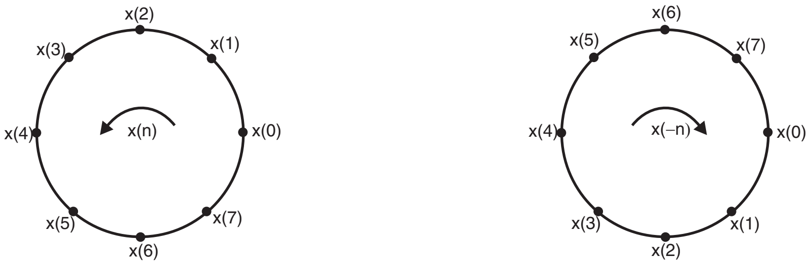


Fig 1.22: *Circular representation of an 8-point sequence and its folded sequence.*

Definition of Circular Convolution

The ***circular convolution*** of two periodic discrete time sequences $x_1(n)$ and $x_2(n)$ with periodicity of N samples is defined as,

$$\boxed{x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N} \quad \text{or} \quad \boxed{x_3(n) = \sum_{m=0}^{N-1} x_2(m)\, x_1((n-m))_N} \qquad .....(1.17)$$

where, $x_3(n)$ = The sequence obtained by circular convolution,

$x_1((n-m))_N$ = Circular shift of $x_1(n)$

$x_2((n-m))_N$ = Circular shift of $x_2(n)$

m is a dummy variable.

The output sequence $x_3(n)$ obtained by circular convolution is also a periodic sequence with periodicity of N samples. Hence, this convolution is also called ***periodic convolution***.

The convolution relation of equation (1.17) can be symbolically expressed as,

$$x_3(n) = x_1(n) \circledast x_2(n) = x_2(n) \circledast x_1(n) \qquad .....(1.18)$$

where, the symbol ⊛ indicates the circular convolution operation.

Circular convolution is defined for periodic sequences, but it can be performed with nonperiodic sequences by periodically extending them. The circular convolution of two sequences requires that, at least one of the sequences be periodic. Hence, it is sufficient if one of the sequences is periodically extended in order to perform circular convolution.

Circular convolution of finite duration sequences can be performed only if both the sequences consist of the same number of samples. If the sequences have different number of samples, then convert the smaller size sequence to the length of the larger size sequence by appending zeros.

Circular convolution basically involves the same four steps as that for linear convolution, namely, folding one sequence, shifting the folded sequence, multiplying the two sequences and finally summing the values of the product sequence. Like linear convolution, any one of the sequence is folded and rotated in circular convolution.

The difference between the two is that in circular convolution the folding and shifting (rotating) operations are performed in a circular fashion by computing the index of one of the sequences by modulo-N operation. In linear convolution there is no modulo-N operation.

Procedure for Evaluating Circular Convolution

Let $x_1(n)$ and $x_2(n)$ be periodic discrete time sequences with a periodicity of N-samples. If $x_1(n)$ and $x_2(n)$ are nonperiodic, then convert the sequences to N-sample sequences and periodically extend the sequence $x_2(n)$ with periodicity of N-samples.

Now the circular convolution of $x_1(n)$ and $x_2(n)$ will produce a periodic sequence $x_3(n)$ with periodicity of N-samples. The samples of one period of $x_3(n)$ can be computed using equation (1.17). The value of $x_3(n)$ at n = q is obtained by replacing n by q, in equation (1.17).

$$\therefore\ x_3(q) = \sum_{m=0}^{N-1} x_1(m)\, x_2((q-m))_N \qquad .....(1.19)$$

The evaluation of equation (1.19) to determine the value of $x_3(n)$ at n = q involves the following five steps.

1. Change of index : Change the index n in the sequences $x_1(n)$ and $x_2(n)$, in order to get the sequences $x_1(m)$ and $x_2(m)$. Represent the samples of one period of the sequences on circles.

2. Folding : Fold $x_2(m)$ about m = 0, to obtain $x_2(-m)$.

3. Rotation : Rotate $x_2(-m)$ by q times in anti-clockwise direction if q is positive and Rotate $x_2(-m)$ by q times in clockwise direction if q is negative to obtain $x_2((q-m))_N$.

4. Multiplication : Multiply $x_1(m)$ by $x_2((q-m))_N$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x_1(m)\, x_2((q-m))_N$.

5. Summation : Sum up the samples of one period of the product sequence $v_q(m)$ to obtain the value of $x_3(n)$ at n = q. [i.e., $x_3(q)$].

The above procedure will give the value of $x_3(n)$ at a single time instant, say $n = q$. In general we are interested in evaluating the values of the sequence $x_3(n)$ in the range $0 < n < N-1$. Hence the steps 3, 4 and 5 given above must be repeated for all possible time shifts in the range $0 < n < N-1$.

Linear Convolution via Circular Convolution

When two numbers of N-point sequences are circularly convolved, it produces another N-point sequence. For circular convolution, one of the sequence should be periodically extended. Also the resultant sequence is periodic with period N.

The linear convolution of two sequences of length N_1 and N_2 produces an output sequence of length $N_1 + N_2 - 1$. To perform linear convolution via circular convolution both the sequences should be converted to $N_1 + N_2 -1$ point sequences by padding with zeros. Then perform circular convolution of $N_1 + N_2 -1$ point sequences. The resultant sequence will be the same as that of linear convolution of N_1 and N_2 point sequences.

Circular Convolution Tabular Method

Let $x_1(n)$ and $x_2(n)$ be the given N-point sequences. Let $x_3(n)$ be the N-point sequence obtained by circular convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to obtain one sample of $x_3(n)$ at $n = q$.

1. Change the index n in the sequences $x_1(n)$ and $x_2(n)$ to get $x_1(m)$ and $x_2(m)$ and the represent the sequences as two rows of tabular array.
2. Fold one of the sequence. Let us fold $x_2(m)$ to get $x_2(-m)$.
3. Periodically extend $x_2(-m)$. Here the periodicity is N, where N is the length of the give sequences.
4. Shift the sequence $x_2(-m)$, q times to get the sequence $x_2((q-m))_N$. If q is positive, then shift the sequence to the right and if q is negative then shift the sequence to the left.
5. Determine the product sequence $x_1(m)\, x_{2,q}(m)$ for one period where, $x_{2,q}(m) = x_2((q-m))_N$
6. The sum of all the samples of the product sequence gives the sample $x_3(q)$ [i.e., $x_3(n)$ at $n = q$].

   Therefore, the sample of $x_3(q)$ at $n = q$ is given by,

$$x_3(q) = \sum_{m=0}^{N-1} x_1(m)\, x_2((q-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,q}(m)$$

The above procedure is repeated for all possible values of n to get the sequence $x_3(n)$.

Example 1.3

Find the linear and circular convolution of the sequences, $x(n) = \{\underset{\uparrow}{1}, 0.5\}$ and $h(n) = \{\underset{\uparrow}{0.5}, 1\}$.

Solution

Linear Convolution by Tabular Array

Let, $y(n) = x(n) * h(n)$

By convolution sum formula,

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\, h(n-m)\ ,\ \text{where m is a dummy variable for convolution.}$$

Since both x(n) and h(n) start at n = 0, the output sequence y(n) will also start at n = 0.

Since the length of x(n) and h(n) is 2, the length of y(n) is $2 + 2 - 1 = 3$.

Let us change the index n to m in x(n) and h(n). The sequences x(m) and h(m) are represented in the tabular array as shown below.

Note: *The unfilled boxes in the table are considered as zeros.*

m	–1	0	1	2
x(m)		1	0.5	
h(m)		0.5	1	
$h(-m) = h_0(m)$	1	0.5		
$h(1-m) = h_1(m)$		1	0.5	
$h(2-m) = h_2(m)$			1	0.5

Each sample of y(n) is given by the equation,

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\,h(n-m) = \sum_{m=-\infty}^{\infty} x(m)\,h_n(m) \;;\; \text{where } h_n(m) = h(n-m)$$

$$\text{When } n = 0 \;;\; y(0) = \sum_{m=-\infty}^{\infty} x(m)\,h(-m) = \sum_{m=-1}^{1} x(m)\,h_0(m) = x(-1)\,h_0(-1) + x(0)\,h_0(0) + x(1)\,h_0(1)$$

$$= 0\times1 + 1\times0.5 + 0.5\times0 = 0 + 0.5 + 0 = 0.5$$

$$\text{When } n = 1 \;;\; y(1) = \sum_{m=-\infty}^{\infty} x(m)\,h(1-m) = \sum_{m=0}^{1} x(m)\,h_1(m) = 1 + 0.25 = 1.25$$

$$\text{When } n = 2 \;;\; y(2) = \sum_{m=-\infty}^{\infty} x(m)\,h(2-m) = \sum_{m=0}^{2} x(m)\,h_2(m) = 0 + 0.5 + 0 = 0.5$$

$$\therefore\ y(n) = \{\underset{\uparrow}{0.5},\ 1.25,\ 0.5\}$$

Circular Convolution by Tabular Array

Let , $y(n) = x(n) \circledast h(n)$

By definition of circular convolution,

$$y(n) = \sum_{m=0}^{N-1} x(m)\ h((n-m))_N \;;\; \text{where m is a dummy variable for convolution.}$$

The index n in the sequences are changed to m and the sequences are represented in the tabular array as shown below. The shifted sequence $h_n(m)$ is periodically extended with periodicity N = 2.

Note: *The bold faced number is the sample obtained by periodic extension.*

m	–1	0	1
x(m)		1	0.5
h(m)		0.5	1
$h((-m))_2 = h_0(m)$	1	0.5	**1**
$h((1-m))_2 = h_1(m)$		1	0.5

Each sample of y(n) is given by the relation,

$$y(n) = \sum_{m=0}^{N-1} x(m)\,h((n-m))_N = \sum_{m=0}^{N-1} x(m)\,h_n(m) \;;\; \text{where } h_n(m) = h((n-m))_N$$

$$\text{When } n = 0 \;;\; y(0) = \sum_{m=0}^{N-1} x(m)\,h((0-m))_2 = \sum_{m=0}^{1} x(m)\,h_0(m)$$

$$= x(0)\,h_0(0) + x(1)\,h_0(1) = 1 \times 0.5 + 0.5 \times 1 = 0.5 + 0.5 = 1.0$$

$$\text{When } n = 1 \; ; \; y(1) \; = \sum_{m=0}^{N-1} x(m)\, h((1-m))_2 \; = \sum_{m=0}^{1} x(m)\, h_1(m)$$

$$= x(0)\, h_1(0) + x(1)\, h_1(1) = 1 \times 1 \; + \; 0.5 \times 0.5 = 1 + 0.25 = 1.25$$

$$\therefore \; y(n) = \{\underset{\uparrow}{1.0}, \; 1.25\}$$

1.1.15 Sectioned Convolution

The response of an LTI system for any arbitrary input is given by linear convolution of the input and the impulse response of the system. If one of the sequences (either the input sequence or impulse response sequence) is very much larger than the other, then it is very difficult to compute the linear convolution for the following reasons.

1. The entire sequence should be available before convolution can be carried out. This involves a long delay in getting the output.
2. Large amount of memory is required to store the sequences.

The above problems can be overcome in sectioned convolutions. In this technique the larger sequence is sectioned (or split) into the size of the smaller sequence. Then the linear convolution of each section of the longer sequence and the smaller sequence is performed.

The output sequences obtained from the convolutions of all the sections are combined to get the overall output sequence. There are two methods of sectioned convolutions. They are overlap add method and overlap save method.

Overlap Add Method

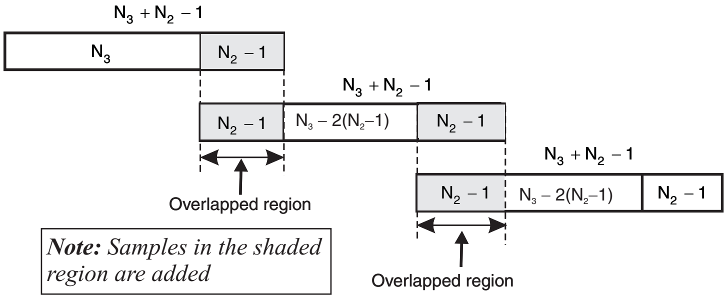
In the ***overlap add method,*** the longer sequence is divided into smaller sequences. Then linear convolution of each section of the longer sequence and the smaller sequence is performed. The overall output sequence is obtained by combining the output of the sectioned convolution.

Let, N_1 = Length of longer sequence

N_2 = Length of smaller sequence

Let the longer sequence be divided into sections of size N_3 samples.

Note: *Normally the longer sequence is divided into sections of size same as that of the smaller sequence.*



Note: *Samples in the shaded region are added*

Fig 1.23: *Overlapping of output sequence of sectioned convolution by overlap add method.*

The linear convolution of each section with smaller sequence will produce an output sequence of size $N_3 + N_2 - 1$ samples. In this method the last $N_2 - 1$ samples of each output sequence overlap with the first $N_2 - 1$ samples of the next section [i.e., there will be a region of $N_2 - 1$ samples over which the output sequence of q^{th} convolution overlaps with the output sequence of $(q + 1)^{th}$ convolution]. While combining the output sequences of the various sectioned convolutions, the corresponding samples of overlapped regions are added and the samples of non-overlapped regions are retained as such.

Overlap Save Method

In the ***overlap save method,*** the results of linear convolution of the various sections are obtained using circular convolution. In this method, the longer sequence is divided into smaller sequences. Each section of the longer sequence and the smaller sequence is converted to the size of the output sequence of sectioned convolution.

The circular convolution of each section of the longer sequence and the smaller sequence is performed. The overall output sequence is obtained by combining the outputs of the sectioned convolution.

Let, N_1 = Length of longer sequence

N_2 = Length of smaller sequence

Let the longer sequence be divided into sections of size N_3 samples.

Note: *Normally the longer sequence is divided into sections of size same as that of the smaller sequence.*

In the ***overlap save method,*** the results of linear convolution are obtained by circular convolution. Hence, each section of the longer sequence and the smaller sequence is converted to the size of the output sequence of size $N_3 + N_2 - 1$ samples.

The smaller sequence is converted to the size of $N_3 + N_2 - 1$ samples, by appending with zeros. The conversion of each section of the longer sequence to the size $N_3 + N_2 - 1$ samples can be performed by two different methods.

Method-1

In this method, the first $N_2 - 1$ samples of a section are appended as last $N_2 - 1$ samples of the previous section [i.e., the overlapping samples are placed at the beginning of the section]. The circular convolution of each section will produce an output sequence of size $N_3 + N_2 - 1$ samples. In this output, the first $N_2 - 1$ samples are discarded and the remaining samples of the output of sectioned convolutions are saved as the overall output sequence.

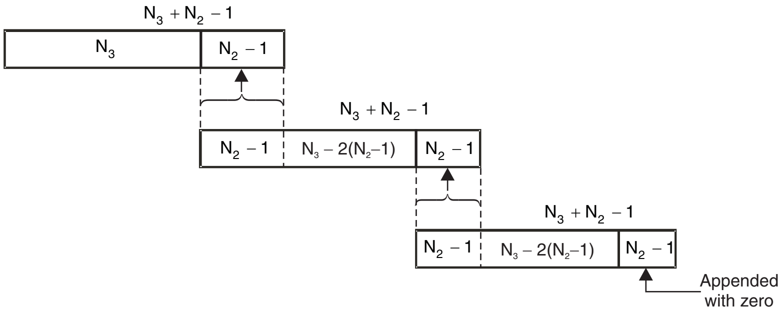
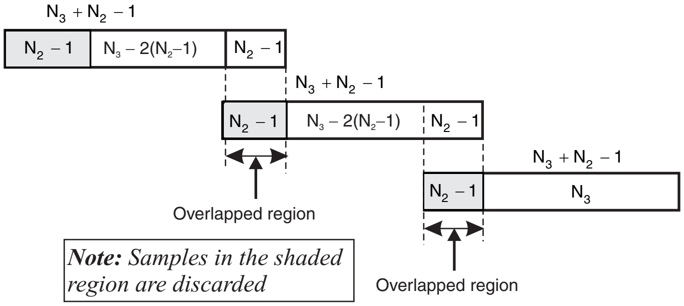


Fig 1.24: *Appending of sections of input sequence in method-1 of overlap save method.*



***Note:** Samples in the shaded region are discarded*

***Fig 1.25:** Overlapping of output sequence of sectioned convolution by method-1 of overlap save method.*

Method-2

In this method, the last N_2–1 samples of a section are appended as last N_2 –1 samples of the next section (i.e, the overlapping samples are placed at the end of the sections). The circular convolution of each section will produce an output sequence of size $N_3 + N_2$ –1 samples.

In this output, the last N_2 –1 samples are discarded and the remaining samples of the output of sectioned convolutions are saved as the overall output sequence.

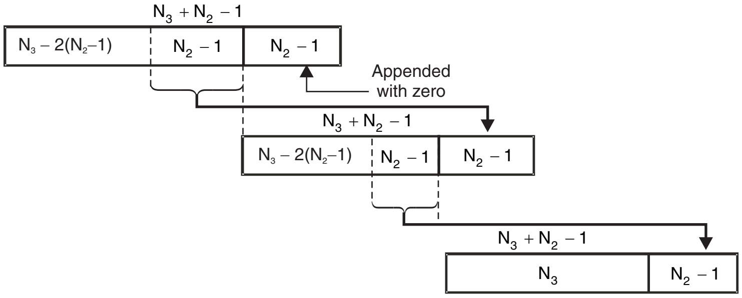
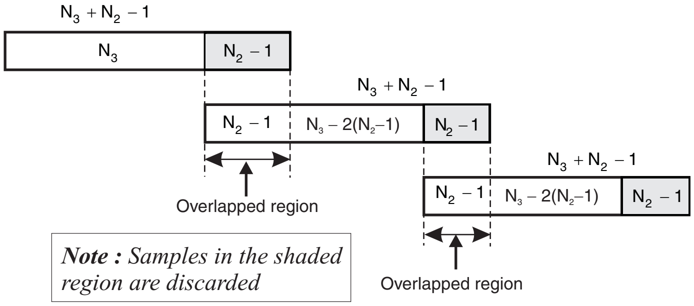


***Fig 1.26:** Appending of sections of input sequence in method-2 of overlap save method.*



***Note :** Samples in the shaded region are discarded*

***Fig 1.27:** Overlapping of output sequence of sectioned convolution by method-2 of overlap save method.*

Example 1.4

Perform the linear convolution of the following sequences by: a) overlap add method and b) overlap save method.

$$x(n) = \{1, -1, 2, -2, 3, -3, 4, -4\} \ ; \ h(n) = \{-1, 1\}$$

Solution

a) Overlap Add Method

In this method the longer sequence is sectioned into sequences of size equal to the smaller sequence. Here x(n) is a longer sequence when compared to h(n). Hence, x(n) is sectioned into sequences of size equal to h(n).

Given that, $x(n) = \{1, -1, 2, -2, 3, -3, 4, -4\}$

Let x(n) can be sectioned into four sequences, each consisting of two samples of x(n) as shown below.

$x_1(n) = 1 \ ; \ n = 0$	$x_2(n) = 2 \ ; \ n = 2$	$x_3(n) = 3 \ ; \ n = 4$	$x_4(n) = 4 \ ; \ n = 6$
$= -1 \ ; \ n = 1$	$= -2 \ ; \ n = 3$	$= -3 \ ; \ n = 5$	$= -4 \ ; \ n = 7$

Let $y_1(n)$, $y_2(n)$, $y_3(n)$ and $y_4(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$, $x_3(n)$ and $x_4(n)$ with h(n), respectively.

Here h(n) starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $\quad\therefore\ y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 2$, $\quad\therefore\ y_2(n)$ will start at $n = n_2 + n_h = 2 + 0 = 2$

$x_3(n)$ starts at $n = n_3 = 4$, $\quad\therefore\ y_3(n)$ will start at $n = n_3 + n_h = 4 + 0 = 4$

$x_4(n)$ starts at $n = n_4 = 6$, $\quad\therefore\ y_4(n)$ will start at $n = n_4 + n_h = 6 + 0 = 6$

Here linear convolution of each section is performed between two sequences, each consisting of 2 samples. Hence, each convolution output will consists of 2 + 2 – 1 = 3 samples. The convolution of each section is performed by a tabular method as shown below.

> ***Note:***
>
> 1. *Here $N_1 = 8$, $N_2 = 2$, $N_3 = 2$. $\quad N_2 - 1 = 2 - 1 = 1 \quad$ and $\quad N_2 + N_3 - 1 = 2 + 2 - 1 = 3$*
> 2. *The unfilled boxes in the tables are considered as zero.*
> 3. *For convenience of convolution operation the index n is replaced by m in $x_1(n)$, $x_2(n)$, $x_3(n)$, $x_4(n)$ and $h(n)$.*

Convolution of Section 1

m	–1	0	1	2
$x_1(m)$		1	–1	
$h(m)$		–1	1	
$h(-m) = h_o(m)$	1	–1		
$h(1 - m) = h_1(m)$		1	–1	
$h(2 - m) = h_2(m)$			1	–1

$$y_1(n) = x_1(n) * h(n) = \sum_{m=-\infty}^{\infty} x_1(m)\, h(n-m)$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\, h_n(m) \ ; \ n = 0, 1, 2$$

$$\text{When } n = 0 \ ; \ y_1(0) = \sum x_1(m)\, h_0(m) = 0 - 1 + 0 = -1$$

$$\text{When } n = 1 \ ; \ y_1(1) = \sum x_1(m)\, h_1(m) = 1 + 1 = 2$$

$$\text{When } n = 2 \ ; \ y_1(2) = \sum x_1(m)\, h_2(m) = 0 - 1 + 0 = -1$$

$$\therefore y_1(n) = \{\underset{\underset{n=0}{\uparrow}}{-1}, 2, -1\}$$

Convolution of Section 2

m	–1	0	1	2	3	4
$x_2(m)$				2	–2	
h(m)		–1	1			
h(–m)	1	–1				
$h(2-m) = h_2(m)$			1	–1		
$h(3-m) = h_3(m)$				1	–1	
$h(4-m) = h_4(m)$					1	–1

$$y_2(n) = x_2(n) * h(n) = \sum_{m=-\infty}^{\infty} x_2(m)\, h(n-m)$$

$$= \sum_{m=-\infty}^{\infty} x_2(m)\, h_n(m)\ ;\ n = 2,3,4$$

When n = 2 ; $y_2(2) = \sum x_2(m)\, h_2(m) = 0 - 2 + 0 = -2$

When n = 3 ; $y_2(3) = \sum x_2(m)\, h_3(m) = 2 + 2 \quad = \quad 4$

When n = 4 ; $y_2(4) = \sum x_2(m)\, h_4(m) = 0 - 2 + 0 = -2$

$$\therefore y_2(n) = \{\underset{\underset{n=2}{\uparrow}}{-2}, 4, -2\}$$

Convolution of Section 3

m	–1	0	1	2	3	4	5	6
$x_3(m)$						3	–3	
h(m)		–1	1					
h(–m)	1	–1						
$h(4-m) = h_4(m)$					1	–1		
$h(5-m) = h_5(m)$						1	–1	
$h(6-m) = h_6(m)$							1	–1

$$y_3(n) = x_3(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_3(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x_3(m)\, h_n(m)\ ;\ n = 4, 5, 6$$

where, $h_n(m) = h(n-m)$

When n = 4 ; $y_3(4) = \sum x_3(m)\, h_4(m) = 0 - 3 + 0 = -3$

When n = 5 ; $y_3(5) = \sum x_3(m)\, h_5(m) = 3 + 3 \quad = \quad 6$

When n = 6 ; $y_3(6) = \sum x_3(m)\, h_6(m) = 0 - 3 + 0 = -3$

$$\therefore y_3(n) = \{\underset{\underset{n=4}{\uparrow}}{-3}, 6, -3\}$$

Convolution of Section 4

m	–1	0	1	2	3	4	5	6	7	8
$x_4(m)$								4	–4	
h(m)		–1	1							
h(–m)	1	–1								
$h(6-m) = h_6(m)$							1	–1		
$h(7-m) = h_7(m)$								1	–1	
$h(8-m) = h_8(m)$									1	–1

$$y_4(n) = x_4(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_4(m)\, h(n-m) \;=\; \sum_{m=-\infty}^{+\infty} x_4(m)\, h_n(m) \;\; ; \;\; n \;=\; 6, 7, 8$$

where, $h_n(m) = h(n-m)$

$$\text{When } n = 6\, ;\, y_4(6) = \sum x_4(m)\, h_6(m) = 0 - 4 + 0 = -4$$
$$\text{When } n = 7\, ;\, y_4(7) = \sum x_4(m)\, h_7(m) = 4 + 4 \quad = \; 8$$
$$\text{When } n = 8\, ;\, y_4(8) = \sum x_4(m)\, h_8(m) = 0 - 4 + 0 = -4$$

$$\therefore y_4(n) = \{\underset{\uparrow}{-4},\; 8, -4\}$$

To Combine the Output of Convolution of Each Section

It can be observed that the last sample in an output sequence overlaps with the first sample of the next output sequence. In this method the overall output is obtained by combining the outputs of convolution of all the sections. The overlapped portions (or samples) are added while combining the output. The output of all the sections can be represented in a table as shown below. Then the samples corresponding to the same value of n are added to get the overall output.

n	0	1	2	3	4	5	6	7	8
$y_1(n)$	–1	2	–1						
$y_2(n)$			–2	4	–2				
$y_3(n)$					–3	6	–3		
$y_4(n)$							–4	8	–4
y(n)	–1	2	–3	4	–5	6	–7	8	–4

$$\therefore \;\; y(n) = x(n) * h(n) = \{-1, 2, -3, 4, -5, 6, -7, 8, -4\}$$

b) Overlap Save Method

In this method, the longer sequence is sectioned into sequences of size equal to the smaller sequence. The number of samples that will be obtained in the output of linear convolution of each section is determined. Then each section of the longer sequence is converted to the size of output sequence using the samples of the original longer sequence. The smaller sequence is also converted to the size of output sequence by appending with zeros. Then the circular convolution of each section is performed.

Here x(n) is a longer sequence when compared to h(n). Hence, x(n) is sectioned into sequences of size equal to h(n). Given that, x(n) = {1, –1, 2, –2, 3, –3, 4, –4}

Let x(n) be sectioned into four sequences, each consisting of two samples of x(n) as shown below.

$x_1(n) = \;\; 1\, ; \;\; n = 0$	$x_2(n) = \;\; 2\; ; \; n = 2$	$x_3(n) = \;\; 3 \;\; ; \;\; n = 4$	$x_4(n) = \;\; 4 \;\; ; \; n = 6$
$= -1\, ; \;\; n = 1$	$= -2\; ; \; n = 3$	$= -3 \;\; ; \;\; n = 5$	$= -4 \;\; ; \; n = 7$

Let $y_1(n)$, $y_2(n)$, $y_3(n)$ and $y_4(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$, $x_3(n)$ and $x_4(n)$ with h(n), respectively. Here linear convolution of each section will result in an output sequence consisting of $2 + 2 - 1 = 3$ samples.

The sequence h(n) is converted to 3-sample sequence by appending with zeros.

$$\therefore\; h(n) = \{-1, 1, 0\}$$

Method-1

In method 1, the overlapping samples are placed at the beginning of the sections. Each section of the longer sequence is converted to 3-sample sequences, using the samples of the original longer sequence as shown below. It can be observed that the first sample of $x_2(n)$ is placed as the overlapping sample at the end of $x_1(n)$. The first sample of $x_3(n)$ is placed as the overlapping sample at the end of $x_2(n)$. The first sample of $x_4(n)$ is placed as the overlapping sample at the end of $x_3(n)$. Since there is no fifth section, the overlapping sample of $x_4(n)$ is taken as zero.

$$\begin{array}{llllllll}
x_1(n) = 1\,; & n=0 & x_2(n) = \textcircled{2}\,; & n=2 & x_3(n) = \textcircled{3}\,; & n=4 & x_4(n) = \textcircled{4}\,; & n=6 \\
\quad = -1\,; & n=1 & \quad = -2\,; & n=3 & \quad = -3\,; & n=5 & \quad = -4\,; & n=7 \\
\quad = 2\,; & n=2 & \quad = 3\,; & n=4 & \quad = 4\,; & n=6 & \quad = 0\,; & n=8
\end{array}$$

Now perform circular convolution of each section with h(n). The output sequence obtained from circular convolution will have three samples. The circular convolution of each section is performed by the tabular method as shown below.

Here, h(n) starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $\therefore$ $y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 2$, $\therefore$ $y_2(n)$ will start at $n = n_2 + n_h = 2 + 0 = 2$

$x_3(n)$ starts at $n = n_3 = 4$, $\therefore$ $y_3(n)$ will start at $n = n_3 + n_h = 4 + 0 = 4$

$x_4(n)$ starts at $n = n_4 = 6$, $\therefore$ $y_4(n)$ will start at $n = n_4 + n_h = 6 + 0 = 6$

Note: *1. Here, $N_1 = 8, N_2 = 2, N_3 = 2$. $\therefore$ $N_2 - 1 = 2 - 1 = 1$ and $N_2 + N_3 - 1 = 2 + 2 - 1 = 3$*

2. The bold faced numbers in the tables are obtained by periodic extension.

3. For convenience of convolution operation, the index n in $x_1(n)$, $x_2(n)$, $x_3(n)$, $x_4(n)$ and h(n) are replaced by m.

Convolution of Section 1

m	–2	–1	0	1	2
$x_1(m)$			1	–1	2
h(m)			–1	1	0
$h((0-m))_3 = h_0(m)$	0	1	–1	**0**	**1**
$h((1-m))_3 = h_1(m)$		0	1	–1	**0**
$h((2-m))_3 = h_2(m)$			0	1	–1

$$y_1(n) = x_1(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_1(m)\, h((n-m))_N$$

$$= \sum_{m=0}^{2} x_1(m)\, h_n(m)\,;\ n = 0, 1, 2$$

where, $h_n(m) = h((n-m))_N$

When n = 0 ; $y_1(0) = \sum x_1(m)\, h_0(m) = -1 + 0 + 2 = 1$

When n = 1 ; $y_1(1) = \sum x_1(m)\, h_1(m) = 1 + 1 + 0 = 2$

When n = 2 ; $y_1(2) = \sum x_1(m)\, h_2(m) = 0 - 1 - 2 = -3$

$$\therefore y_1(n) = \{\underset{\underset{n=0}{\uparrow}}{1},\ 2, -3\}$$

Convolution of Section 2

m	–2	–1	0	1	2	3	4
$x_2(m)$					2	–2	3
h(m)			–1	1	0		
h(–m)	0	1	–1				
$h((2-m))_3 = h_2(m)$			0	1	–1	**0**	**1**
$h((3-m))_3 = h_3(m)$				0	1	–1	**0**
$h((4-m))_3 = h_4(m)$					0	1	–1

$$y_2(n) = x_2(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_2(m)\, h\big((n-m)\big)_N = \sum_{m=2}^{4} x_2(m)\, h_n(m) \;\; ; \;\; n \; = \; 2, 3, 4$$

$$\text{where, } h_n(m) = h((n-m))_N$$

When $n = 2$; $y_2(2) = \sum x_2(m)\, h_2(m) = -2 + 0 + 3 = 1$

When $n = 3$; $y_2(3) = \sum x_2(m)\, h_3(m) = 2 + 2 + 0 = 4$

When $n = 4$; $y_2(4) = \sum x_2(m)\, h_4(m) = 0 - 2 - 3 = -5$

$$\therefore y_2(n) = \{\underset{\substack{\uparrow \\ n=2}}{1},\; 4, -5\}$$

Convolution of Section 3

m	–2	–1	0	1	2	3	4	5	6
$x_3(m)$							3	–3	4
h(m)			–1	1	0				
h(–m)	0	1	–1						
$h((4-m))_3 = h_4(m)$					0	1	–1	**0**	**1**
$h((5-m))_3 = h_5(m)$						0	1	–1	**0**
$h((6-m))_3 = h_6(m)$							0	1	–1

$$y_3(n) = x_3(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_3(m)\, h\big((n-m)\big)_N = \sum_{m=4}^{6} x_3(m)\, h_n(m) \, ; n \; = \; 4, 5, 6$$

$$\text{where, } h_n(m) = h((n-m))_N$$

When $n = 4$; $y_3(4) = \sum x_3(m)\, h_4(m) = -3 + 0 + 4 = 1$

When $n = 5$; $y_3(5) = \sum x_3(m)\, h_5(m) = 3 + 3 + 0 = 6$

When $n = 6$; $y_3(6) = \sum x_3(m)\, h_6(m) = 0 - 3 - 4 = -7$

$$\therefore y_3(n) = \{\underset{\substack{\uparrow \\ n=4}}{1},\; 6, -7\}$$

Convolution of section 4

m	–2	–1	0	1	2	3	4	5	6	7	8
$x_4(m)$									4	–4	0
h(m)			–1	1	0						
h(–m)	0	1	–1								
$h((6-m))_3 = h_6(m)$							0	1	–1	**0**	**1**
$h((7-m))_3 = h_7(m)$								0	1	–1	**0**
$h((8-m))_3 = h_8(m)$									0	1	–1

$$y_4(n) = x_4(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_4(m)\, h((n-m))_N = \sum_{m=6}^{8} x_4(m)\, h_n(m)\ ;\ n\ =\ 6, 7, 8$$

where, $h_n(m) = h((n-m))_N$

When $n = 6\ ;\ y_4(6) = \sum x_4(m)\, h_6(m) = -4 + 0 + 0 = -4$

When $n = 7\ ;\ y_4(7) = \sum x_4(m)\, h_7(m) = \quad 4 + 4 + 0 = \quad 8$

When $n = 8\ ;\ y_4(8) = \sum x_4(m)\, h_8(m) = \quad 0 - 4 + 0 = -4$

$$\therefore\ y_4(n) = \{\underset{\substack{\uparrow \\ n=6}}{-4},\ 8, -4\}$$

To Combine the Output of the Convolution of Each Section

It can be observed that the last sample in an output sequence overlaps with the first sample of the next output sequence. In overlap save method, the overall output is obtained by combining the outputs of the convolution of all the sections. While combining the outputs, the overlapped first sample of every output sequence is discarded and the remaining samples are simply saved as samples of y(n) as shown in the following table.

n	0	1	2	3	4	5	6	7	8
$y_1(n)$	$\cancel{1}$	2	−3						
$y_2(n)$			$\cancel{1}$	4	−5				
$y_3(n)$					$\cancel{1}$	6	−7		
$y_4(n)$							$\cancel{-4}$	8	−4
y(n)	×	2	−3	4	−5	6	−7	8	−4

$y(n) = x(n) * h(n) = \{\times,\ 2, -3, -4, -5, 6, -7, 8, -4\}$

> ***Note:*** *Here y(n) is linear convolution of x(n) and h(n). It can be observed that the results of both the methods are same, except the first $N_2 - 1$ samples.*

Method-2

In method-2, the overlapping samples are placed at the end of the section. Each section of the longer sequence is converted to 3-sample sequence, using the samples of the original longer sequence as shown below. It can be observed that the last sample of $x_1(n)$ is placed as the overlapping sample at the end of $x_2(n)$.

The last sample of $x_2(n)$ is placed as the overlapping sample at the end of $x_3(n)$. The last sample of $x_3(n)$ is placed as the overlapping sample at the end of $x_4(n)$. Since there is no previous section for $x_1(n)$, the overlapping sample of $x_1(n)$ is taken as zero.

$$\begin{array}{llll}
x_1(n) = \ \ 1\ ;\ n=0 & x_2(n) = \ \ 2\ ;\ n=2 & x_3(n) = \ \ 3\ ;\ n=4 & x_4(n) = \ \ 4\ ;\ n=6 \\
\qquad = -1\ ;\ n=1 & \qquad = -2\ ;\ n=3 & \qquad = -3\ ;\ n=5 & \qquad = -4\ ;\ n=7 \\
\qquad = \ \ 0\ ;\ n=2 & \qquad = -1\ ;\ n=4 & \qquad = -2\ ;\ n=6 & \qquad = -3\ ;\ n=8
\end{array}$$

Now perform circular convolution of each section with h(n). The output sequence obtained from circular convolution will have three samples. The circular convolution of each section is performed by the tabular method as shown ahead.

Here, $h(n)$ starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $\therefore$ $y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$
$x_2(n)$ starts at $n = n_2 = 2$, $\therefore$ $y_2(n)$ will start at $n = n_2 + n_h = 2 + 0 = 2$
$x_3(n)$ starts at $n = n_3 = 4$, $\therefore$ $y_3(n)$ will start at $n = n_3 + n_h = 4 + 0 = 4$
$x_4(n)$ starts at $n = n_4 = 6$, $\therefore$ $y_4(n)$ will start at $n = n_4 + n_h = 6 + 0 = 6$

> ***Note:*** *1. Here $N_1 = 8, N_2 = 2, N_3 = 2.$ $\therefore N_2 - 1 = 2 - 1 = 1$ and $N_2 + N_3 - 1 = 2 + 2 - 1 = 3$*
> *2. The bold faced numbers in the tables are obtained by periodic extension.*
> *3. For convenience of convolution, the index n is replaced by m in $x_1(n)$, $x_2(n)$, $x_3(n)$, $x_4(n)$ and $h(n)$.*

Convolution of Section 1

m	–2	–1	0	1	2
$x_1(m)$			1	–1	0
$h(m)$			–1	1	0
$h((-m))_3 = h_0(m)$	0	1	–1	**0**	**1**
$h((1-m))_3 = h_1(m)$		0	1	–1	**0**
$h((2-m))_3 = h_2(m)$			0	1	–1

$$y_1(n) = x_1(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_1(m)\, h((n-m))_N$$

$$= \sum_{m=0}^{2} x_1(m)\, h_n(m)\,;\ n = 0,1,2$$

where, $h_n(m) = h((n-m))_N$

When $n = 0\,;\ y_1(0) = \sum x_1(m)\, h_0(m) = -1 + 0 + 0 = -1$

When $n = 1\,;\ y_1(1) = \sum x_1(m)\, h_1(m) = 1 + 1 + 0 = 2$

When $n = 2\,;\ y_1(2) = \sum x_1(m)\, h_2(m) = 0 - 1 + 0 = -1$

$$\therefore\ y_1 = \{\underset{\substack{\uparrow \\ n=0}}{-1}, 2, -1\}$$

Convolution of Section 2

m	–2	–1	0	1	2	3	4
$x_2(m)$					2	–2	–1
$h(m)$			–1	1	0		
$h(-m)$	0	1	–1				
$h((2-m))_3 = h_2(m)$			0	1	–1	**0**	**1**
$h((3-m))_3 = h_3(m)$				0	1	–1	**0**
$h((4-m))_3 = h_4(m)$					0	1	–1

$$y_2(n) = x_2(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_2(m)\, h((n-m))_N = \sum_{m=2}^{4} x_2(m)\, h_n(m)\,;\ n = 2,3,4\ ;\ \text{where, } h_n(m) = h((n-m))_N$$

When $n = 2\,;\ y_2(2) = \sum x_2(m)\, h_2(m) = -2 + 0 - 1 = -3$

When $n = 3\,;\ y_2(3) = \sum x_2(m)\, h_3(m) = 2 + 2 + 0 = 4$

When $n = 4\,;\ y_2(4) = \sum x_2(m)\, h_4(m) = 0 - 2 + 1 = -1$

$$\therefore\ y_2(n) = \{\underset{\substack{\uparrow \\ n=2}}{-3}, 4, -1\}$$

Convolution of Section 3

m	–2	–1	0	1	2	3	4	5	6
$x_3(m)$							3	–3	–2
$h(m)$			–1	1	0				
$h(-m)$	0	1	–1						
$h((4-m))_3 = h_4(m)$					0	1	–1	**0**	**1**
$h((5-m))_3 = h_5(m)$						0	1	–1	**0**
$h((6-m))_3 = h_6(m)$							0	1	–1

$$y_3(n) = x_3(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_3(m)\, h((n-m))_N = \sum_{m=4}^{6} x_3(m)\, h_n(m)\ ;\ n = 4, 5, 6$$

where, $h_n(m) = h((n-m))_N$

When n = 4 ; $y_3(4) = \sum x_3(m)\, h_4(m) = -3 + 0 - 2 = -5$

When n = 5 ; $y_3(5) = \sum x_3(m)\, h_5(m) = 3 + 3 + 0 = 6$

When n = 6 ; $y_3(6) = \sum x_3(m)\, h_6(m) = 0 - 3 + 2 = -1$

$$\therefore y_3(n) = \{\underset{\underset{n=4}{\uparrow}}{-5},\ 6, 1\}$$

Convolution of Section 4

m	–2	–1	0	1	2	3	4	5	6	7	8
$x_4(m)$									4	–4	–3
$h(m)$			–1	1	0						
$h(-m)$	0	1	–1								
$h((6-m))_3 = h_6(m)$							0	1	–1	**0**	**1**
$h((7-m))_3 = h_7(m)$								0	1	–1	**0**
$h((8-m))_3 = h_8(m)$									0	1	–1

$$y_4(n) = x_4(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_4(m)\, h((n-m))_N = \sum_{m=6}^{8} x_4(m)\, h_n(m)\ ;\ n = 6, 7, 8$$

where, $h_n(m) = h((n-m))_N$

When n = 6 ; $y_4(6) = \sum x_4(m)\, h_6(m) = -4 + 0 - 3 = -7$

When n = 7 ; $y_4(7) = \sum x_4(m)\, h_7(m) = 4 + 4 + 0 = 8$

When n = 8 ; $y_4(8) = \sum x_4(m)\, h_8(m) = 0 - 4 + 3 = -1$

$$\therefore y_4(n) = \{\underset{\underset{n=6}{\uparrow}}{-7},\ 8, -1\}$$

To Combine the Output of the Convolution of Each Section

It can be observed that the last sample in an output sequence overlaps with the first sample of the next output sequence. In overlap save method, the overall output is obtained by combining the outputs of the convolution of all the sections. While combining the outputs, the overlapped last sample of every output sequence is discarded and the remaining samples are simply saved as samples of y(n) as shown in the following table.

n	0	1	2	3	4	5	6	7	8
$y_1(n)$	−1	2	$\xcancel{-1}$						
$y_2(n)$			−3	4	$\xcancel{-1}$				
$y_3(n)$					−5	6	$\xcancel{-1}$		
$y_4(n)$							−7	8	$\xcancel{-1}$
y(n)	−1	2	−3	4	−5	6	−7	8	×

Note:

Here y(n) is linear convolution of x(n) and h(n). It can be observed that the results of both the methods are same except the last $N_2 - 1$ samples.

$$\therefore\ y(n) = x(n) * h(n) = \{-1, 2, -3, 4, -5, 6, -7, 8, \times\}$$

Example 1.5

Perform the linear convolution of the following sequences by: a) overlap add method and b) overlap save method.

$$x(n) = \{1, 2, 3, -1, -2, -3, 4, 5, 6\} \text{ and } h(n) = \{2, 1, -1\}$$

Solution

a) Overlap Add Method

In this method the longer sequence is sectioned into sequences of size equal to the smaller sequence. Here x(n) is a longer sequence when compared to h(n). Hence, x(n) is sectioned into sequences of size equal to h(n).

Given that, x(n) = {1, 2, 3, −1, −2, −3, 4, 5, 6}.

Let x(n) be sectioned into three sequences, each consisting of three samples of x(n) as shown below.

$x_1(n) = 1\ ;\ n = 0$	$x_2(n) = -1\ ;\ n = 3$	$x_3(n) = 4\ ;\ n = 6$
$= 2\ ;\ n = 1$	$= -2\ ;\ n = 4$	$= 5\ ;\ n = 7$
$= 3\ ;\ n = 2$	$= -3\ ;\ n = 5$	$= 6\ ;\ n = 8$

Let $y_1(n)$, $y_2(n)$ and $y_3(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$ and $x_3(n)$ with h(n), respectively.

Here h(n) starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $\therefore$ $y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 3$, $\therefore$ $y_2(n)$ will start at $n = n_2 + n_h = 3 + 0 = 3$

$x_3(n)$ starts at $n = n_3 = 6$, $\therefore$ $y_3(n)$ will start at $n = n_3 + n_h = 6 + 0 = 6$

Here linear convolution of each section is performed between two sequences each consisting of three samples. Hence, each convolution output will consist of 3 + 3 − 1 = 5 samples. The convolution of each section is performed by the tabular method shown ahead:

Note: *1. Here, $N_1 = 9, N_2 = 3, N_3 = 3, \quad \therefore N_2 - 1 = 3 - 1 = 2$ and $N_2 + N_3 - 1 = 3 + 3 - 1 = 5$.*

2. The unfilled boxes in the table are considered as zero.

3. For convenience of convolution operation, the index n is replaced by m in $x_1(n)$, $x_2(n)$, $x_3(n)$ and $h(n)$.

Convolution of Section 1

m	–2	–1	0	1	2	3	4
$x_1(m)$			1	2	3		
$h(m)$			2	1	–1		
$h(-m) = h_0(m)$	–1	1	2				
$h(1 - m) = h_1(m)$		–1	1	2			
$h(2 - m) = h_2(m)$			–1	1	2		
$h(3 - m) = h_3(m)$				–1	1	2	
$h(4 - m) = h_4(m)$					–1	1	2

$$y_1(n) = x_1(n) * h(n) = \sum_{m=-\infty}^{\infty} x_1(m)\, h(n-m)$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\, h_n(m) \; ; n = 0, 1, 2, 3, 4$$

where, $h_n(m) = h(n-m)$

$$\therefore y_1(n) = \{\underset{\substack{\uparrow \\ n=0}}{2}, 5, 7, -1, -3\}$$

When n = 0 ; $y_1(0) = \Sigma\, x_1(m)\, h_o(m) = 0 + 0 + 2 + 0 + 0 = 2$

When n = 1 ; $y_1(1) = \Sigma\, x_1(m)\, h_1(m) = 0 + 1 + 4 + 0 = 5$

When n = 2 ; $y_1(2) = \Sigma\, x_1(m)\, h_2(m) = -1 + 2 + 6 = 7$

When n = 3 ; $y_1(3) = \Sigma\, x_1(m)\, h_3(m) = 0 - 2 + 3 + 0 = 1$

When n = 4 ; $y_1(4) = \Sigma\, x_1(m)\, h_4(m) = 0 + 0 - 3 + 0 + 0 = -3$

Convolution of Section 2

m	–2	–1	0	1	2	3	4	5	6	7
$x_2(m)$						–1	–2	–3		
$h(m)$			2	1	–1					
$h(-m) = h_0(m)$	–1	1	2							
$h(3 - m) = h_3(m)$				–1	1	2				
$h(4 - m) = h_4(m)$					–1	1	2			
$h(5 - m) = h_5(m)$						–1	1	2		
$h(6 - m) = h_6(m)$							–1	1	2	
$h(7 - m) = h_7(m)$								–1	1	2

$$y_2(n) = x_2(n) * h(n) = \sum_{m=-\infty}^{\infty} x_2(m)\, h(n-m) = \sum_{m=-\infty}^{\infty} x_2(m)\, h_n(m) \; ; \; n = 3, 4, 5, 6, 7$$

where, $h_n(m) = h(n-m)$

When n = 3 ; $y_2(3) = \Sigma\, x_2(m)\, h_3(m) = 0 + 0 - 2 + 0 + 0 = -2$

When n = 4 ; $y_2(4) = \Sigma\, x_2(m)\, h_4(m) = 0 - 1 - 4 + 0 = -5$

When n = 5 ; $y_2(5) = \Sigma\, x_2(m)\, h_5(m) = 1 - 2 - 6 = -7$

When n = 6 ; $y_2(6) = \Sigma\, x_2(m)\, h_6(m) = 0 + 2 - 3 + 0 = -1$

When n = 7 ; $y_2(7) = \Sigma\, x_2(m)\, h_7(m) = 0 + 0 + 3 + 0 + 0 = 3$

$$\therefore y_2(n) = \{\underset{\substack{\uparrow \\ n=3}}{-2}, -5, -7, -1, 3\}$$

Convolution of Section 3

m	–2	–1	0	1	2	3	4	5	6	7	8	9	10
$x_3(m)$									4	5	6		
$h(m)$			2	1	–1								
$h(-m) = h_0(m)$	–1	1	2										
$h(6-m) = h_6(m)$							–1	1	2				
$h(7-m) = h_7(m)$								–1	1	2			
$h(8-m) = h_8(m)$									–1	1	2		
$h(9-m) = h_9(m)$										–1	1	2	
$h(10-m) = h_{10}(m)$											–1	1	2

$$y_3(n) = x_3(n) * h(n) = \sum_{m=-\infty}^{\infty} x_3(m)\,h(n-m) = \sum_{m=-\infty}^{\infty} x_3(m)\,h_n(m) \;;\; n = 6, 7, 8, 9, 10$$

where, $h_n(m) = h(n-m)$

When n = 6 ; $y_3(6) = \Sigma\, x_3(m)\,h_6(m) = 0 + 0 + 8 + 0 + 0 = 8$

When n = 7 ; $y_3(7) = \Sigma\, x_3(m)\,h_7(m) = 0 + 4 + 10 + 0 = 14$

When n = 8 ; $y_3(8) = \Sigma\, x_3(m)\,h_8(m) = -4 + 5 + 12 = 13$

When n = 9 ; $y_3(9) = \Sigma\, x_3(m)\,h_9(m) = 0 - 5 + 6 + 0 = 1$

When n = 10 ; $y_3(10) = \Sigma\, x_3(m)\,h_{10}(m) = 0 + 0 - 6 + 0 + 0 = -6$

$$\therefore\ y_2(n) = \{\underset{\substack{\uparrow\\ n=6}}{8}, 14, 13, 1, -6\}$$

To Combine the Output of the Convolution of each Section

It can be observed that the last $N_2 - 1$ sample in an output sequence overlaps with the first $N_2 - 1$ sample of the next output sequence. In this method, the overall output is obtained by combining the outputs of convolution of all the sections. The overlapped portions (or samples) are added while combining the output.

The output of all the sections can be presented in a table as shown below. Then the samples corresponding to the same value of n are added to get the overall output.

n	0	1	2	3	4	5	6	7	8	9	10
$y_1(n)$	2	5	7	1	–3						
$y_2(n)$				–2	–5	–7	–1	3			
$y_3(n)$							8	14	13	1	–6
$y(n)$	2	5	7	–1	–8	–7	7	17	13	1	–6

$\therefore\ y(n) = x(n) * h(n) = \{2, 5, 7, -1, -8, -7, 7, 17, 13, 1, -6\}$

---

b) Overlap Save Method

In this method the longer sequence is sectioned into sequences of size equal to the smaller sequence. The number of samples that will be obtained in the output of linear convolution of each section is determined. Then each section of the longer sequence is converted to the size of the output sequence using the samples of the original longer sequences. The smaller sequence is also converted to the size of output sequence by appending with zeros. Then the circular convolution of each section is performed.

Here, x(n) is a longer sequence when compared to h(n). Hence, x(n) is sectioned into sequences of size equal to h(n).

Given that $x(n) = \{1, 2, 3, -1, -2, -3, 4, 5, 6\}$.

Let x(n) be sectioned into three sequences each consisting of three samples as shown below:

Let , N_1 = Length of longer sequence

N_2 = Length of smaller sequence

$N_3 = N_2$ = Length of each section of longer sequence.

$x_1(n) = 1 \,;\, n = 0$	$x_2(n) = -1 \,;\, n = 3$	$x_3(n) = 4 \,;\, n = 6$
$= 2 \,;\, n = 1$	$= -2 \,;\, n = 4$	$= 5 \,;\, n = 7$
$= 3 \,;\, n = 2$	$= -3 \,;\, n = 5$	$= 6 \,;\, n = 8$

Let $y_1(n)$, $y_2(n)$ and $y_3(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$ and $x_3(n)$ with h(n) respectively. Here linear convolution of each section will result in an output sequence consisting of 3 + 3 – 1 = 5 samples.

Hence each section of longer sequence is converted to five sample sequence, using the samples of the original longer sequence as shown below. It can be observed that the first $N_2 - 1$ samples of $x_2(n)$ is placed as overlapping sample at the end of $x_1(n)$. The first $N_2 - 1$ samples of $x_3(n)$ is placed as overlapping sample at the end of $x_2(n)$. Since there is no fourth section, the overlapping samples of $x_3(n)$ are considered as zeros.

$x_1(n) = 1 \,;\, n = 0$	$x_2(n) = -1 \,;\, n = 3$	$x_3(n) = 4 \,;\, n = 6$
$= 2 \,;\, n = 1$	$= -2 \,;\, n = 4$	$= 5 \,;\, n = 7$
$= 3 \,;\, n = 2$	$= -3 \,;\, n = 5$	$= 6 \,;\, n = 8$
$= -1 \,;\, n = 3$	$= 4 \,;\, n = 6$	$= 0 \,;\, n = 9$
$= -2 \,;\, n = 4$	$= 5 \,;\, n = 7$	$= 0 \,;\, n = 10$

The sequence h(n) is also converted to a five sample sequence by appending with zeros.

$\therefore$ h(n) = {2, 1, –1, 0, 0}

Now perform circular convolution of each section with h(n). The output sequence obtained from circular convolution will have five samples. The circular convolution of each section is performed by the tabular method shown below.

Here h(n) starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $\therefore$ $y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 3$, $\therefore$ $y_2(n)$ will start at $n = n_2 + n_h = 3 + 0 = 3$

$x_3(n)$ starts at $n = n_3 = 6$, $\therefore$ $y_3(n)$ will start at $n = n_3 + n_h = 6 + 0 = 6$

Note: *1. Here $N_1 = 9$, $N_2 = 3$, $N_3 = 3$ $\therefore$ $N_2 - 1 = 3 - 1 = 2$ and $N_2 + N_3 - 1 = 3 + 3 - 1 = 5$ samples.*

2. The bold faced numbers in the table are obtained by periodic extension.

3. For convenience of convolution operation, the index n is replaced by m in $x_1(n)$, $x_2(n)$, $x_3(n)$ and h(n).

Convolution of Section 1

m	–4	–3	–2	–1	0	1	2	3	4
$x_1(m)$					1	2	3	–1	–2
h(m)					2	1	–1	0	0
$h((-m))_5 = h_0(m)$	0	0	–1	1	2	**0**	**0**	**–1**	**1**
$h((1-m))_5 = h_1(m)$		0	0	–1	1	2	**0**	**0**	**–1**
$h((2-m))_5 = h_2(m)$			0	0	–1	1	2	**0**	**0**
$h((3-m))_5 = h_3(m)$				0	0	–1	1	2	**0**
$h((4-m))_5 = h_3(m)$					0	0	–1	1	2

$$y_1(n) = x_1(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_1(m)\, h\big((n-m)\big)_N = \sum_{m=0}^{4} x_1(m)\, h_n(m)\ ;\ n\ =\ 0, 1, 2, 3, 4$$

where, $h_n(m) = h((n-m))_N$

When $n = 0$; $y_1(0) = \Sigma\, x_1(m)h_o(m) = 2 + 0 + 0 + 1 - 2 = 1$

When $n = 1$; $y_1(0) = \Sigma\, x_1(m)h_1(m) = 1 + 4 + 0 + 0 + 2 = 7$

When $n = 2$; $y_1(2) = \Sigma\, x_1(m)h_2(m) = -1 + 2 + 6 + 0 + 0 = 7$

When $n = 3$; $y_1(3) = \Sigma\, x_1(m)h_3(m) = 0 - 2 + 3 - 2 + 0 = -1$

When $n = 4$; $y_1(4) = \Sigma\, x_1(m)h_4(m) = 0 + 0 - 3 - 1 - 4 = -8$

$$\therefore y_1(n) = \{\underset{\substack{\uparrow \\ n=0}}{1}, 7, 7, -1, -8\}$$

Convolution of Section 2

m	−4	−3	−2	−1	0	1	2	3	4	5	6	7
$x_2(m)$								−1	−2	−3	4	5
$h(m)$					2	1	−1	0	0			
$h(-m) = h_0(m)$	0	0	−1	1	2							
$h((3-m))_5 = h_3(m)$				0	0	−1	1	2	**0**	**0**	**−1**	**1**
$h((4-m))_5 = h_4(m)$					0	0	−1	1	2	**0**	**0**	**−1**
$h((5-m))_5 = h_5(m)$						0	0	−1	1	2	**0**	**0**
$h((6-m))_5 = h_6(m)$							0	0	−1	1	2	**0**
$h((7-m))_5 = h_7(m)$								0	0	−1	1	2

$$y_2(n) = x_2(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_2(m)\, h\big((n-m)\big)_N = \sum_{m=3}^{7} x_2(m)\, h_n(m)\ \ ;\ n\ =\ 3, 4, 5, 6, 7$$

where, $h_n(m) = h((n-m))_N$

When $n = 3$; $y_2(3) = \Sigma\, x_2(m)h_3(m) = -2 + 0 + 0 - 4 + 5 = -1$

When $n = 4$; $y_2(4) = \Sigma\, x_2(m)h_4(m) = -1 - 4 + 0 + 0 - 5 = -10$

When $n = 5$; $y_2(5) = \Sigma\, x_2(m)h_5(m) = 1 - 2 - 6 + 0 + 0 = -7$

When $n = 6$; $y_2(6) = \Sigma\, x_2(m)h_6(m) = 0 + 2 - 3 + 8 + 0 = 7$

When $n = 7$; $y_2(7) = \Sigma\, x_2(m)h_7(m) = 0 + 0 + 3 + 4 + 10 = 17$

$$\therefore y_1(n) = \{\underset{\substack{\uparrow \\ n=3}}{-1}, -10, -7, 7, 17\}$$

Convolution of Section 3

m	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10
$x_3(m)$											4	5	6	0	0
$h(m)$					2	1	−1	0	0						
$h(-m) = h_0(m)$	0	0	−1	1	2										
$h((6-m))_5 = h_6(m)$							0	0	−1	1	2	**0**	**0**	**−1**	**1**
$h((7-m))_5 = h_7(m)$								0	0	−1	1	2	**0**	**0**	**−1**
$h((8-m))_5 = h_8(m)$									0	0	−1	1	2	**0**	**0**
$h((9-m))_5 = h_9(m)$										0	0	−1	1	2	**0**
$h((10-m))_5 = h_{10}(m)$											0	0	−1	1	2

$$y_3(n) = x_3(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_3(m)\, h((n-m))_N = \sum_{m=6}^{10} x_3(m)\, h_n(m) \;\; ; \;\; n = 6, 7, 8, 9, 10$$

where, $h_n(m) = h((n-m))_N$

When n = 6 ; $y_3(6) = \Sigma\, x_3(m)h_6(m) = 8 + 0 + 0 + 0 + 0 = 8$

When n = 7 ; $y_3(7) = \Sigma\, x_3(m)h_7(m) = 4 + 10 + 0 + 0 + 0 = 14$

When n = 8 ; $y_3(8) = \Sigma\, x_3(m)h_8(m) = -4 + 5 + 12 + 0 + 0 = 13$

When n = 9 ; $y_3(9) = \Sigma\, x_3(m)h_9(m) = 0 - 5 + 6 + 0 + 0 = 1$

When n = 10 ; $y_3(10) = \Sigma\, x_3(m)h_{10}(m) = 0 + 0 - 6 + 0 + 0 = -6$

$$\therefore\; y_1(n) = \{\underset{\substack{\uparrow \\ n=6}}{8}, 14, 13, 1, -6\}$$

To Combine the Output of Convolution of Each Section

It can be observed that the last N_2-1 samples in an output sequence overlaps with the first N_2-1 samples of next the output sequence. In overlap save method, the overall output is obtained by combining the outputs of convolution of all the sections. While combining the outputs, the overlapped first N_2-1 samples of every output sequence is discarded and the remaining samples are simply saved as samples of y(n) as shown in the following table.

n	0	1	2	3	4	5	6	7	8	9	10
$y_1(n)$	~~1~~	~~7~~	7	−1	−8						
$y_2(n)$				~~−1~~	~~10~~	−7	7	17			
$y_3(n)$							~~8~~	~~14~~	13	1	−6
y(n)	×	×	7	−1	−8	−7	7	17	13	1	−6

$\therefore$ y(n) = x(n) ∗ h(n) = { ×, ×, 7, −1, −8, −7, 7, 17, 13, 1, −6}

> ***Note:*** *Here y(n) is linear convolution of x(n) and h(n). It can be observed that the results of both the methods are same except the first $N_2 - 1$ samples.*

1.2 Concept of Frequency in Signals

1.2.1 Concept of Frequency in Continuous Time Signals

In the waveform of periodic continuous time signals, the waveshape repeats with respect to time. The number of identical waveshape or pattern in one second is called ***frequency***. A pattern of waveform is called ***cycle***. Therefore, the unit of frequency is *cycles per second* or *Hertz(Hz)*. The time for one cycle of waveform is called ***period***. The unit of period is *seconds*. Therefore, the concept of frequency is directly related to concept of time and frequency has the dimension of inverse time.

The complex exponential continuous time signal is the generalized periodic signal.

The complex exponential signal is defined as,

$$x(t) = Ae^{j(\Omega t + \phi)} = Ae^{j(2\pi F t + \phi)}$$

where,

A = Amplitude

Ω = Angular frequency in *rad/second*

F = Frequency in *cycles/seconds*

$T = \frac{1}{F}$ = Period (or Time period) in *seconds*

ϕ = Phase in radians per *seconds*

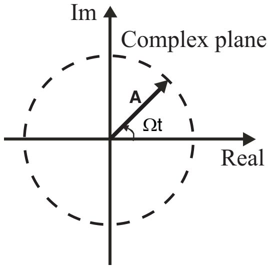


Fig 1.28: *Complex exponential signal.*

The complex exponential signal can be represented in a complex plane by a rotating vector, which rotates with a constant angular velocity of Ω *rad/second*.

The complex exponential signal can be resolved into real and imaginary parts as shown below:

$$\begin{aligned} x(t) &= Ae^{j(\Omega t + \phi)} \\ &= A\cos(\Omega t + \phi) + jA\sin(\Omega t + \phi) \end{aligned}$$

$$\begin{aligned} \therefore\ A\cos(\Omega t + \phi) &= \text{Real part of } x(t) \\ A\sin(\Omega t + \phi) &= \text{Imaginary part of } x(t) \end{aligned}$$

From the above equation, we can say that a complex exponential signal is the vector sum of two sinusoidal signals of the form $\cos\Omega t$ and $\sin\Omega t$.

Sinusoidal Signal

Case (i): Cosinusoidal Signal

The cosinusoidal signal is defined as,

$$x(t) = A\cos(\Omega t + \phi)$$

When $\phi = 0$, $x(t) = A\cos\Omega t$

When ϕ = Positive, $x(t) = A\cos(\Omega t + \phi)$

When ϕ = Negative, $x(t) = A\cos(\Omega t - \phi)$

Case (ii): Sinusoidal Signal

The sinusoidal signal is defined as,

$$x(t) = A\sin(\Omega t + \phi)$$

When $\phi = 0$, $x(t) = A\sin\Omega t$

When ϕ = Positive, $x(t) = A\sin(\Omega t + \phi)$

When ϕ = Negative, $x(t) = A\sin(\Omega t - \phi)$

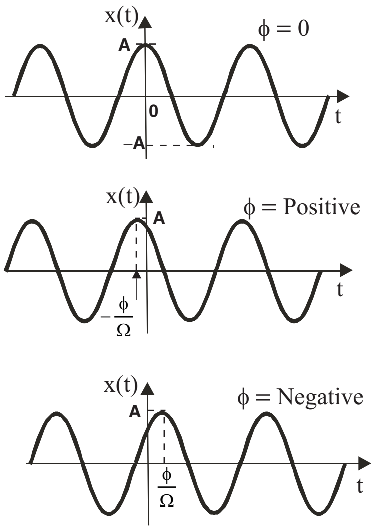


Fig 1.29: *Cosinusoidal signal.*

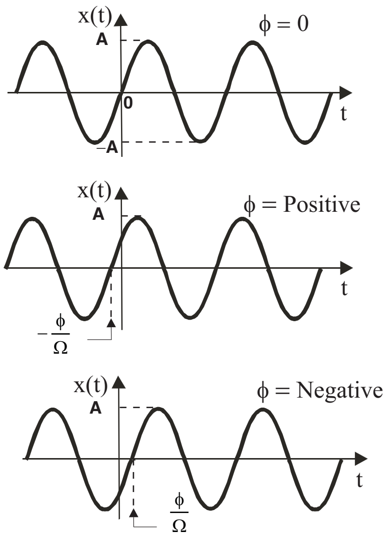


Fig 1.30: *Sinusoidal signal.*

Properties of periodic continuous time signals

1. For any value of F, the periodic signal, x(t) will satisfy the following relation, $x(t + T) = x(t)$.
2. Continuous time periodic time signals with different frequencies are distinct.
3. Sinusoidal signal can be represented by sum of two complex exponential signal with positive and negative frequency as shown below:

   $$A\sin \Omega t = \frac{A}{2j}\left(e^{j\Omega t} - e^{-j\Omega t}\right) \quad ; \quad A\cos \Omega t = \frac{A}{2}\left(e^{j\Omega t} + e^{-j\Omega t}\right)$$

   Therefore, the sinusoids are sum of two rotating phasors/vectors that rotate in opposite directions. The component with positive frequency rotate in anticlockwise directions and the component with negative frequency rotate in clockwise direction.

   Alternatively, when the real sine or cosine signal has to be represented in terms of complex exponential, then a signal with negative frequency is required. Here it should be understood that the signal with negative frequency is not a physically realizable signal, but it is required for mathematical representation of real signals in terms of complex exponential signals.
4. Increase in frequency will result is increase in number of cycles per second. Therefore, frequency can be increased without limit and so range of frequency of continuous time signal is $-\infty$ to $+\infty$ (i.e, $-\infty < F < +\infty$).

1.2.2 Concept of Frequency in Discrete Time Signals

Discrete time signals are sampled version of continuous time signals and uniformally sampled using a sampling time, T_s. During sampling, a part of signal is represented by a sample in a sampling time instant. If there are N samples, in a period then each sample represents $\frac{1}{N}^{th}$ part of the signal in a period. Therefore, the unit of discrete time frequency, f can be expressed as cycles/samples (or more appropriately, fraction of a cycle/sample). Therefore, the unit of angular discrete time frequency, ω is radian/sample. Here, $\omega = 2\pi f$.

The discrete time sinusoidal and their properties are discussed is Section 1.1.7.

1.3 Summary of Analysis and Synthesis Equation for FT and DTFT

1.3.1 Development of Fourier Transform from Fourier Series

The exponential form of Fourier series representation of a periodic signal is given by,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n \, e^{jn\Omega_0 t} \qquad .....(1.20)$$

$$\text{where, } c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\Omega_0 t}\, dt \qquad .....(1.21)$$

In the Fourier representation using equation (1.20), the c_n for various values of n are the spectral components of the signal x(t), located at intervals of fundamental frequency Ω_0. Therefore, the frequency spectrum is discrete in nature.

The Fourier representation of a signal using equation (1.20) is applicable for periodic signals. For Fourier representation of nonperiodic signals, let us consider that the fundamental period tends to infinity. When the fundamental period tends to infinity, the fundamental frequency Ω_0 tends to zero or becomes very small. Since fundamental frequency Ω_0 is very small, the spectral components lie very close to each other and so the frequency spectrum becomes continuous.

In order to obtain the Fourier representation of a nonperiodic signal let us consider that the fundamental frequency Ω_0 is very small.

Let, $\Omega_0 \to \Delta\Omega$

On replacing Ω_0 by $\Delta\Omega$ in equation (1.20) we get,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n \, e^{jn\Delta\Omega t}$$

On substituting for c_n in the above equation from equation (1.21) (by taking τ as dummy variable for integration), we get,

$$x(t) = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau} \, d\tau \right] e^{jn\Delta\Omega t} \qquad .....(1.22)$$

We know that, $\Omega_0 = 2\pi F_0 = \frac{2\pi}{T}$; $\therefore \frac{1}{T} = \frac{\Omega_0}{2\pi}$

$$\text{Since } \Omega_0 \to \Delta\Omega, \quad \frac{1}{T} = \frac{\Delta\Omega}{2\pi} \qquad .....(1.23)$$

On substituting for $\frac{1}{T}$ from equation (1.23) in equation (1.22) we get,

$$x(t) = \sum_{n=-\infty}^{+\infty} \left[\frac{\Delta\Omega}{2\pi} \int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau} d\tau \right] e^{jn\Delta\Omega t}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left[\int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau} d\tau \right] e^{jn\Delta\Omega t} \, \Delta\,\Omega$$

For nonperiodic signals, the fundamental period T tends to infinity. On letting limit T tends to infinity in the above equation we get,

$$x(t) = \underset{T\to\infty}{\text{Lt}} \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left[\int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau} \, d\tau \right] e^{jn\Delta\Omega t} \; \Delta\Omega$$

$$\text{When } T \to \infty \; ; \; \sum \to \int \; ; \; \Delta\Omega \to d\Omega$$

$$\therefore \; x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x(\tau)\, e^{-jn\Omega\tau} \, d\tau \right] e^{jn\Omega t} \; d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, e^{jn\Omega t} \; d\Omega \qquad .....(1.24)$$

$$\text{where, } X(j\Omega) = \int_{-\infty}^{+\infty} x(\tau) \;\; e^{-jn\Omega\tau} \, d\tau$$

Since τ is a dummy variable, change τ to t.

$$\therefore \; X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-jn\Omega t} \, dt \qquad ....(1.25)$$

Equation (1.25) is ***Fourier transform of x(t)*** and equation (1.24) is ***inverse Fourier transform of x(t).***

Analysis: *Equation (1.25) extracts the frequency components of the signal, and transformation using equation (1.25) is also called analysis of the signal x(t).*

Synthesis: *Equation (1.24) combines the frequency components of the signal and so the inverse transformation using equation (1.24) is also called synthesis of the signal x(t).*

1.3.2 Development of Discrete Time Fourier Transform from Discrete Time Fourier Series

Let $\tilde{x}(n)$ be a periodic sequence with period N. If the period N tends to infinity then the periodic sequence $\tilde{x}(n)$ will become a nonperiodic sequence x(n).

$$\therefore x(n) = \underset{N\to\infty}{\text{Lt}}\ \tilde{x}(n)$$

Let c_k be Fourier coefficients of $\tilde{x}(n)$.

$$\therefore c_k = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}(n)\,e^{\frac{-j2\pi kn}{N}} \quad\Rightarrow\quad Nc_k = \sum_{n=0}^{N-1}\tilde{x}(n)\,e^{\frac{-j2\pi kn}{N}}$$

Since $\tilde{x}(n)$ is periodic, for even values of N, the summation index in the above equation can be changed from $n =-\left(\frac{N}{2}-1\right)$ to $+\frac{N}{2}$. (For odd values of N, the summation index is $n=\left(-\frac{N-1}{2}\text{ to }+\frac{N-1}{2}\right)$.

Note: *The change of index is necessary to convert the signal to two sided signal.*

$$\therefore Nc_k = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}}\tilde{x}(n)\,e^{\frac{-j2\pi kn}{N}} = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}}\tilde{x}(n)\,e^{-j\omega_k n} \qquad .....(1.26)$$

where, $\omega_k = \frac{2\pi k}{N}$

Let us define Nc_k as a function of $e^{j\omega_k}$.

$$\therefore\ X(e^{j\omega_k}) = Nc_k \qquad .....(1.27)$$

Now, using equation (1.26) equation (1.27) can be expressed as shown below.

$$\therefore\ X(e^{j\omega_k}) = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}}\tilde{x}(n)\,e^{-j\omega_k n} \qquad .....(1.28)$$

Let, $N \to \infty$, in equation (1.28).

Now $\tilde{x}(n) \to x(n)$, $\omega_k \to \omega$, and the summation index becomes $-\infty$ to $+\infty$.

Therefore, the equation (1.28) can be written as shown below:

$$\therefore\ X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\,e^{-j\omega n} \qquad ....(1.29)$$

Equation (1.29) is called **Fourier transform** of x(n), which is used to represent nonperiodic discrete time signal (as a function of frequency, ω) in the frequency domain.

Consider the Fourier series representation of $\tilde{x}(n)$ given below:

$$\tilde{x}(n) = \sum_{k=0}^{N-1} c_k\,e^{\frac{j2\pi kn}{N}}$$

Let us multiply and divide the above equation by $\frac{N}{2\pi}$,

$$\tilde{x}(n) = \frac{N}{2\pi} \times \frac{2\pi}{N} \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi kn}{N}}$$

$$= \frac{1}{2\pi} \sum_{k=0}^{N-1} Nc_k e^{j\omega_k n} \frac{2\pi}{N} \qquad \boxed{\omega_k = \frac{2\pi k}{N}}$$

$$= \frac{1}{2\pi} \sum_{k=0}^{N-1} X(e^{j\omega_k}) e^{j\omega_k n} \frac{2\pi}{N} \qquad .....(1.30) \qquad \boxed{\text{Using equation (1.27).}}$$

Let $N \to \infty$, in equation (1.30).

Now, $\tilde{x}(n) \to x(n)$, $\omega_k \to \omega$, $2\pi / N \to d\omega$, and summation becomes integral with limits 0 to 2π. Therefore, the equation (1.30) can be written as shown below.

$$x(n) = \frac{1}{2\pi} \int_0^{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega \qquad .....(1.31)$$

Equation (1.31) is called **inverse Fourier transform** of x(n), which is used to extract the discrete time signal from its frequency domain representation.

Analysis: *Since equation (1.29) extracts the frequency components of discrete time signal, the transformation using equation (1.29) is also called analysis of discrete time signal x(n).*

Synthesis: *Since equation (1.31) integrates or combines the frequency components of discrete time signal, the inverse transformation using equation (1.31) is also called synthesis of discrete time signal x(n).*

1.4 Frequency Domain Sampling

The discrete time Fourier transform of a non-periodic discrete time signal, x(n) is given by,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(1.32)$$

Here, $X(e^{j\omega})$ is a complex function of discrete time frequency, ω and both magnitude and phase of $X(e^{j\omega})$ is periodic function of ω, with periodicity of $\omega = 0$ to 2π (or $\omega = -\pi$ to π).

Let us sample $X(e^{j\omega})$ at N equally spaced frequency intervals in one period of $X(e^{j\omega})$.

The N equally spaced frequency intervals can be obtained by replacing ω by $\frac{2\pi k}{N}$ for k = 0, 1, 2, ..., N – 1 in the period, $\omega = 0$ to 2π.

Let, $\omega_k = \frac{2\pi k}{N}$ for k = 0, 1, 2, ..., N – 1

Therefore, the frequency sampled version of $X(e^{j\omega})$ can be expressed as shown in equation (1.33).

$$X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N} = \omega_k} = X(e^{j\omega_k}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-\frac{j2\pi kn}{N}} \qquad .....(1.33)$$

The equation (1.33) consists of infinite number of summation of N-point frequency domain sequences for values of n in the range,

n = mN to mN + N – 1, where m = $-\infty$ to $+\infty$

When, $m = -2$, $n = -2N$ to $-2N + N - 1 = -2N$ to $-N - 1$

When, $m = -1$, $n = -N$ to $-N + N - 1 = -N$ to -1

When, $m = 0$, $n = 0$ to $N - 1$

When, $m = 1$, $n = N$ to $N + N - 1 = N$ to $2N - 1$

When, $m = 2$, $n = 2N$ to $2N + N - 1 = 2N$ to $3N - 1$

Therefore, equation (1.33) can be expressed as shown below:

$$X(e^{j\omega_k}) = \ldots.. + \sum_{n=-2N}^{-N-1} x(n)\, e^{-\frac{j2\pi nk}{N}} + \sum_{n=-N}^{-1} x(n)\, e^{-\frac{j2\pi nk}{N}} + \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi nk}{N}}$$

$$+ \sum_{n=N}^{2N-1} x(n)\, e^{-\frac{j2\pi nk}{N}} + \sum_{n=2N}^{3N-1} x(n)\, e^{-\frac{j2\pi nk}{N}} + \ldots\ldots.$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{n=mN}^{mN+N-1} x(n)\, e^{-\frac{j2\pi nk}{N}}$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{n=0}^{N-1} x(n - mN)\, e^{-\frac{j2\pi(n-mN)k}{N}}$$

Let, $n = n - mN$
$\therefore mN = 0$

$$= \sum_{m=-\infty}^{+\infty} \sum_{n=0}^{N-1} x(n - mN)\, e^{-\frac{j2\pi nk}{N}}\, e^{j2\pi mk}$$

Since m and k are integres
$e^{j2\pi mk} = 1.$

$$= \sum_{m=-\infty}^{+\infty} \sum_{n=0}^{N-1} x(n - mN)\, e^{-\frac{j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} \sum_{m=-\infty}^{+\infty} x(n - mN)\, e^{-\frac{j2\pi nk}{N}}$$

Interchanging order of summation.

$$= \sum_{n=0}^{N-1} x_p(n)\, e^{-\frac{j2\pi nk}{N}} \qquad \ldots..(1.34)$$

where, $x_p(n) = \sum_{m=-\infty}^{+\infty} x(n - mN)$

The signal $x_p(n)$ is periodic extension of x(n) with period N samples.

Reconstruction of the periodic signal $x_p(n)$

The Fourier series representation of $x_p(n)$ is given by,

$$x_p(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} \quad ; \quad n = 0, 1, 2, \ldots.., N - 1 \qquad \ldots..(1.35)$$

where, c_k = Fourier coefficient

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-\frac{j2\pi kn}{N}} \quad ; \quad k = 0, 1, 2, \ldots.., N - 1$$

$$= \frac{1}{N} X(e^{j\omega_k}) \qquad \ldots.(1.36)$$

Using equation (1.34).

Using equation (1.36) in equation (1.35) the Fourier series representation of $x_p(n)$ is given by,

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j\omega_k})\, e^{\frac{j2\pi kn}{N}} \qquad \ldots..(1.37)$$

The equation (1.37) can be used to recover (or reconstruct) the time domain discrete time signal x(n) from the frequency domain sampled version of the signal $X(e^{j\omega_k})$. The recovered signal $x_p(n)$ is periodic extension of x(n).

The recovered signal $x_p(n)$ will be exactly represent the periodic extension of x(n), only if the frequency domain signal is sampled at sufficient number of frequency intervals. Here, N is the number samples of frequency domain signals. Let, L be the length of original discrete time signal, x(n).

Now, the value of N should be greater than or equal to L in order to avoid time domain aliasing in the reconstructed signal $x_p(n)$. An example of time domain aliasing is shown in Fig 1.31 when value of N is less than L.

Consider the discrete time signal x(n) and its periodic extension $x_p(n)$ shown in Fig 1.31. Here, the length of the sequence, L = 5. The periodic extension of x(n) for various values of N are shown in Figs 1.31b, c and d.

From the periodic extension $x_{p1}(n)$ and $x_{p2}(n)$ the original signal x(n) can be obtained from one period of the periodic extension but in the periodic extension $x_{p3}(n)$, the original signal x(n) cannot be obtained from one period of periodic extension. In Fig 1.31d, the samples of higher value of L appear in the position of lower value of L and this is called ***time aliasing***.

Conclusion: Frequency domain sampling of a discrete time signal of length, L should be made at sufficient number of frequency intervals, N. Such that $N \geq L$ in order to exactly recover the samples of x(n) from the frequency sampled version of the signal. This forms a condition for computing DFT.

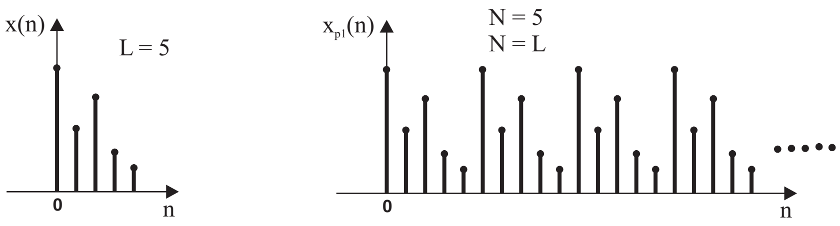


Fig a: *Discrete time signal of length,L.*

Fig b: *Periodic extension of x(n) when N = L.*

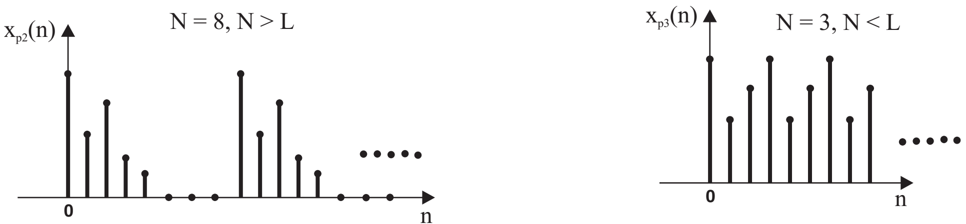


Fig c: *Periodic extension of x(n) when N > L.*

Fig d: *Periodic extension of x(n) when N < L.*

Fig 1.31: *Discrete time signal x(n) and periodic extension $x_p(n)$.*

1.5 Discrete Fourier Transform (DFT)

1.5.1 Deriving DFT from DTFT

The frequency domain representation of a discrete time signal obtained using discrete time Fourier transform (DTFT) will be a continuous and periodic function of ω, with periodicity of 2π. In order to obtain discrete function of ω, the DTFT can be sampled at sufficient number of frequency intervals.

Let, $X(e^{j\omega})$ be discrete time Fourier transform of the discrete time signal x(n). The discrete Fourier transform (DFT) of x(n) is obtained by sampling one period of the discrete time Fourier transform $X(e^{j\omega})$ at a finite number of frequency points.

The frequency domain sampling is conventionally performed at N equally spaced frequency points in the period, $0 \le \omega \le 2\pi$. The sampling frequency points are denoted as ω_k and they are given by,

$$\omega_k = \frac{2\pi k}{N} \quad ; \quad \text{for } k = 0, 1, 2, ..., N-1$$

Now, the DFT is a sequence consisting of N-samples of DTFT. Let the samples be denoted by X(k) for k = 0, 1, 2, ..., N − 1. Therefore, the sampling of $X(e^{j\omega})$ is mathematically expressed as,

$$\boxed{X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \quad ; \quad \text{for } k = 0, 1, 2, ..., N-1} \qquad ...(1.38)$$

The DFT sequence starts at k = 0, corresponding to ω = 0 but does not include k = N, corresponding to ω = 2π, (since the sample at ω = 0 is same as the sample at ω = 2π). Generally, the DFT is defined along with the number of samples and is called ***N-point DFT***. The number of samples N for a finite duration sequence x(n) of length L should be such that N ≥ L, in order to avoid aliasing of the frequency spectrum.

The sampling of Fourier transform of a sequence to get DFT is shown in Example 1.1. To calculate DFT of a sequence it is not necessary to compute the Fourier transform, since the DFT can be directly computed using the definition of DFT as given by equation (1.6).

1.5.2 Definition of Discrete Fourier Transform (DFT)

Let, x(n) = Discrete time signal of length L

X(k) = DFT of x(n)

Now, the ***N-point DFT*** of x(n), where N ≥ L, is defined as,

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad \text{for } k = 0, 1, 2, ..., N-1} \qquad .....(1.39)$$

Symbolically, the N-point DFT of x(n) can be expressed as,

$\mathcal{DFT}\{x(n)\}$

where, $\mathcal{DFT}$ is the operator that represents discrete Fourier transform.

$$\boxed{\therefore \mathcal{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad \text{for } k = 0, 1, 2, ..., N-1}$$

Since X(k) is a sequence consisting of N-complex numbers for k = 0, 1, 2, ..., N − 1, the DFT of x(n) can be expressed as a sequence as shown below:

$$X(k) = \{X(0),\ X(1),\ X(2),\ X(3), ..., X(N-1)\}$$

1.5.3 Frequency Spectrum using DFT

X(k) is a discrete function of frequency of discrete time signal ω and so it is also called ***discrete frequency spectrum*** (or signal spectrum) of the discrete time signal x(n).

X(k) is a complex valued function of k and so it can be expressed in rectangular form as,

$$X(k) = X_r(k) + jX_i(k)$$

where, $X_r(k)$ = Real part of X(k)

$X_i(k)$ = Imaginary part of X(k)

Magnitude function (Magnitude spectrum), |X(k)|: *It is defined as,*

$$\boxed{\text{Magnitude spectrum}, |X(k)| = \sqrt{|X(k)\,X^*(k)}}$$

where $X^*(k)$ is complex conjugate of X(k)

$$\boxed{|X(k)|^2 = X(k)\,X^*(k)}$$

$$\text{Alternatively, } |X(k)|^2 = X(k)X^*(k) = [X_r(k) + jX_i(k)][X_r(k) - jX_i(k)]$$

$$= X_r^2(k) + X_i^2(k)$$

$$\therefore |X(k)| = \sqrt{X_r^2(k) + X_i^2(k)}$$

Phase function (Phase spectrum), ∠X(k): *It is defined as,*

$$\boxed{\text{Phase spectrum}, \angle X(k) = \text{Arg}[X(k)] = \tan^{-1}\left[\frac{X_i(k)}{X_r(k)}\right]}$$

Since X(k) is a sequence consisting of N-complex numbers for k = 0, 1, 2, ..., N − 1, the magnitude and phase spectrum of X(k) can be expressed as a sequence as shown below:

$$\text{Magnitude sequence}, |X(k)| = \{|X(0)|, |X(1)|, |X(2)|, ..., |X(N-1)|\}$$

$$\text{Phase sequence}, \angle X(k) = \{\angle X(0), \angle X(1), \angle X(2), ..., \angle X(N-1)\}$$

The magnitude and phase sequence can be sketched graphically as a function of k.

Magnitude spectrum: *The plot of samples of magnitude sequence versus k is called magnitude spectrum.*

Phase spectrum: *The plot of samples of phase sequence versus k is called phase spectrum.*

Frequency spectrum: *In general, both magnitude and phase spectrum are called frequency spectrum.*

1.5.4 Inverse DFT

Let, x(n) = Discrete time signal

X(k) = N-point DFT of x(n)

The ***inverse DFT*** of the sequence X(k) of length N is defined as,

$$\boxed{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}} \quad ; \quad \text{for } n = 0, 1, 2, ..., N-1} \qquad .....(1.40)$$

Symbolically the inverse DFT of x(n) can be expressed as,

$\mathcal{DFT}^{-1}\{X(k)\}$

where, $\mathcal{DFT}^{-1}$ is the operator that represents inverse DFT.

$$\boxed{\mathcal{DFT}^{-1}\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}} \quad ; \quad \text{for } n = 0, 1, 2, \text{.......}, N-1}$$

We also refer to x(n) and X(k) as a DFT pair and this relation is expressed as,

$$\boxed{x(n) \underset{\mathcal{DFT}^{-1}}{\overset{\mathcal{DFT}}{\rightleftarrows}} X(k)}$$

1.5.5 Relation between DFT and $\mathcal{Z}$-Transform

The $\mathcal{Z}$-transform of N-point sequence x(n) is given by,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}$$

Let us evaluate X(z) at N equally spaced points on unit circle, i.e., at $z = e^{\frac{j2\pi k}{N}}$

> ***Note:*** *Since,* $\left|e^{\frac{j2\pi k}{N}}\right| = 1$ and $\angle e^{\frac{j2\pi k}{N}} = \frac{2\pi k}{N}$,
>
> *the term,* $z = e^{\frac{j2\pi k}{N}}$ *, for k = 0, 1, 2, 3, ..., N– 1*
>
> *represents N equally spaced points on the unit circle in the z-plane.*

$$\therefore \quad X(z)\big|_{z=e^{\frac{j2\pi k}{N}}} = \sum_{n=0}^{N-1} x(n)\, z^{-n}\big|_{z=e^{\frac{j2\pi k}{N}}} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \qquad \text{.....(1.41)}$$

By the definition of N-point DFT we get,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \qquad \text{.....(1.42)}$$

From equations (1.41) and (1.42) we can say that,

$$X(k) = X(z)\Big|_{z=e^{\frac{j2\pi k}{N}}} \qquad \text{.....(1.43)}$$

From equation (1.43), we can conclude that the N-point DFT of a finite duration sequence can be obtained from the $\mathcal{Z}$-transform of the sequence by evaluating the $\mathcal{Z}$-transform of the sequence at N equally spaced points around the unit circle. Since the evaluation is performed on unit circle, the ROC of X(z) should include the unit circle.

1.5.6 Linear Convolution using DFT

Let, x(n) = N_1-point sequence

h(n) = N_2-point sequence

y(n) = Sequence obtained by linear convolution of x(n) and h(n).

The linear convolution is defined as,

$$y(n) = x(n) * h(n) = h(n) * x(n)$$

$$\text{where, } x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m)$$

The DFT supports only circular convolution and so the linear convolution of above equation has to be computed via circular convolution. Since x(n) is N_1-point sequence and h(n) is N_2-point sequence, the linear convolution of x(n) and h(n) will generate y(n) of size $N_1 + N_2 - 1$.

Therefore, in order to perform linear convolution via circular convolution, the x(n) and h(n) should be converted to $N_1 + N_2 - 1$ point sequences by appending zeros. Now the circular convolution of $N_1 + N_2 - 1$ point sequences x(n) and h(n) will give the same result as that obtained by linear convolution.

Example 1.6

Compute 4-point DFT and 8-point DFT of causal three sample sequence given by,

$$x(n) = \frac{1}{3} \; ; \; 0 \le n \le 2$$
$$= 0 \; ; \; \text{else}$$

Solution

By the definition of N-point DFT, the kth complex coefficient of X(k), for $0 \le k \le N-1$, is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$\boxed{x(0) = x(1) = x(2) = \frac{1}{3},\, x(3) = 0}$$

a) 4-point DFT (∴ N = 4)

$$X(k) = \sum_{n=0}^{4-1} x(n)\, e^{\frac{-j2\pi kn}{4}} = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\pi kn}{2}} = x(0)\,e^{0} + x(1)\,e^{\frac{-j\pi k}{2}} + x(2)\,e^{-j\pi k} + x(3)\,e^{\frac{-j\pi k3}{2}}$$

$$= \frac{1}{3} + \frac{1}{3}e^{\frac{-j\pi k}{2}} + \frac{1}{3}e^{-j\pi k} = \frac{1}{3}\left[1 + \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2} + \cos\pi k - j\sin\pi k\right]$$

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

For 4-point DFT, X(k) has to be evaluated for k = 0, 1, 2, 3.

$$\text{When } k = 0\,;\, X(0) = \frac{1}{3}[1 + \cos 0 - j\sin 0 + \cos 0 - j\sin 0]$$
$$= \frac{1}{3}[1 + 1 - j0 + 1 - j0] = 1 = 1\angle 0$$

$$\text{When } k = 1\,;\, X(1) = \frac{1}{3}\left[1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi\right]$$
$$= \frac{1}{3}[1 + 0 - j - 1 - j0] = -j\frac{1}{3} = \frac{1}{3}\angle -\frac{\pi}{2} = 0.333\angle -0.5\pi$$

$$\text{When } k = 2\,;\, X(2) = \frac{1}{3}[1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi]$$
$$= \frac{1}{3}[1 - 1 - j0 + 1 - j0] = \frac{1}{3} = 0.333\angle 0$$

$$\text{When } k = 3\,;\, X(3) = \frac{1}{3}\left[1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi\right]$$
$$= \frac{1}{3}[1 + 0 + j - 1 - j0] = j\frac{1}{3} = \frac{1}{3}\angle\frac{\pi}{2} = 0.333\angle 0.5\pi$$

∴ The 4-point DFT sequence X(k) is given by,

$$X(k) = \{1\angle 0,\ 0.333\angle -0.5\pi,\ 0.333\angle 0,\ 0.333\angle 0.5\pi\}$$

Phase angles are in radians.

$$\therefore \text{ Magnitude Function, } |X(k)| = \{1,\ 0.333,\ 0.333,\ 0.333\}$$

$$\text{Phase Function, } \angle X(k) = \{0,\ -0.5\pi,\ 0,\ 0.5\pi\}$$

b) 8-point DFT (∴N = 8)

$$\boxed{x(0) = x(1) = x(2) = \frac{1}{3},\, x(n) = 0\,;\, n \ge 3}$$

$$X(k) = \sum_{n=0}^{8-1} x(n)\, e^{\frac{-j2\pi kn}{8}} = \sum_{n=0}^{7} x(n)\, e^{\frac{-j\pi kn}{4}} = x(0)\,e^{0} + x(1)\,e^{\frac{-j\pi k}{4}} + x(2)\,e^{\frac{-j\pi k}{2}}$$

$$= \frac{1}{3} + \frac{1}{3}e^{\frac{-j\pi k}{4}} + \frac{1}{3}e^{\frac{-j\pi k}{2}} = \frac{1}{3}\left[1 + \cos\frac{\pi k}{4} - j\sin\frac{\pi k}{4} + \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right]$$

For 8-point DFT, X(k) has to be evaluated for k = 0, 1, 2, 3, 4, 5, 6, 7.

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

$$\text{When } k=0\ ;\ X(0) = \frac{1}{3}\left[1+\cos 0 - j\sin 0 + \cos 0 - j\sin 0\right]$$

$$= \frac{1}{3}(1+1-j0+1-j0) = 1 = 1\angle 0$$

$$\text{When } k=1\ ;\ X(1) = \frac{1}{3}\left[1+\cos\frac{\pi}{4} - j\sin\frac{\pi}{4} + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right]$$

$$= 0.333(1+0.707-j0.707+0-j1)$$

$$\boxed{\frac{0.785}{\pi}\times\pi = 0.25\pi}$$

$$= 0.568 - j0.568 = 0.803\angle -0.785 = 0.803\angle -0.25\pi$$

$$\text{When } k=2\ ;\ X(2) = \frac{1}{3}\left[1+\cos\frac{2\pi}{4} - j\sin\frac{2\pi}{4} + \cos\frac{2\pi}{2} - j\sin\frac{2\pi}{2}\right]$$

$$= 0.333(1+0-j1-1-j0) = -j0.333 = 0.333\angle -\frac{\pi}{2} = 0.333\angle -0.5\pi$$

$$\text{When } k=3\ ;\ X(3) = \frac{1}{3}\left[1+\cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$$

$$= 0.333(1-0.707-j0.707+0+j1) = 0.098+j0.098 = 0.139\angle 0.785 = 0.139\angle 0.25\pi$$

$$\text{When } k=4\ ;\ X(4) = \frac{1}{3}\left[1+\cos\frac{4\pi}{4} - j\sin\frac{4\pi}{4} + \cos\frac{4\pi}{2} - j\sin\frac{4\pi}{2}\right]$$

$$= 0.333(1-1-j0+1-j0) = 0.333 = 0.333\angle 0$$

$$\text{When } k=5\ ;\ X(5) = \frac{1}{3}\left[1+\cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4} + \cos\frac{5\pi}{2} - j\sin\frac{5\pi}{2}\right]$$

$$= 0.333(1-0.707+j0.707+0-j1)$$

$$= 0.098-j0.098 = 0.139\angle -0.785 = 0.139\angle -0.25\pi$$

$$\text{When } k=6\ ;\ X(6) = \frac{1}{3}\left[1+\cos\frac{6\pi}{4} - j\sin\frac{6\pi}{4} + \cos\frac{6\pi}{2} - j\sin\frac{6\pi}{2}\right]$$

$$= 0.333(1+0+j1-1-j0)$$

$$= j0.333 = 0.333\angle\frac{\pi}{2} = 0.333\angle 0.5\pi$$

$$\text{When } k=7\ ;\ X(7) = \frac{1}{3}\left[1+\cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4} + \cos\frac{7\pi}{2} - j\sin\frac{7\pi}{2}\right]$$

$$= 0.333(1+0.707+j0.707+0+j1)$$

Phase angles are in radians.

$$= 0.568+j0.568 = 0.803\angle 0.785 = 0.803\angle 0.25\pi$$

∴ The 8-point DFT sequence X(k) is given by,

$$X(k) = \{1\angle 0,\ 0.803\angle -0.25\pi,\ 0.333\angle -0.5\pi,\ 0.139\angle 0.25\pi,\ 0.333\angle 0,\ 0.139\angle -0.25\pi,\ 0.333\angle 0.5\pi, 0.803\angle 0.25\pi\}$$

$$\therefore \text{ Magnitude Function, } |X(k)| = \{1,\ 0.803,\ 0.333,\ 0.139,\ 0.333,\ 0.139,\ 0.333,\ 0.803\}$$

$$\text{Phase Function, } \angle X(k) = \{0,\ -0.25\pi,\ -0.5\pi,\ 0.25\pi,\ 0,\ -0.25\pi,\ 0.5\pi,\ 0.25\pi\}$$

The magnitude spectrum of X(k) is shown in Figs 1, 2 and 3 for N = 4, N = 8 and N = 16, respectively. The curve shown in a dotted line is the sketch of magnitude function of $X(e^{j\omega})$ for ω in the range 0 to 2π. Here it is observed that the magnitude of DFT coefficients are samples of magnitude function of $X(e^{j\omega})$.

The phase spectrum of X(k) is shown in Figs 4, 5 and 6 for N = 4, N = 8 and N = 16, respectively. The curve shown in a dotted line is the sketch of phase function of $X(e^{j\omega})$ for ω in the range 0 to 2π. Here it is observed that the phase of the DFT coefficients are samples of phase function of $X(e^{j\omega})$.

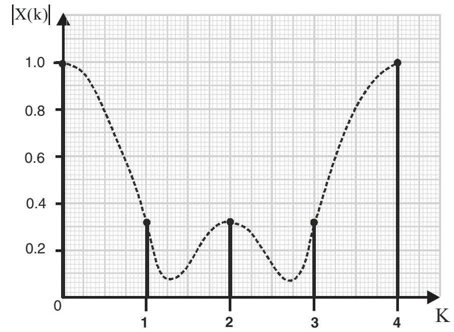


Fig 1: *Magnitude spectrum of X(k) for N=4.*

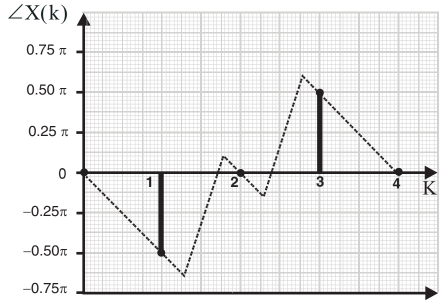


Fig 4: *Phase spectrum of X(k) for N=4.*

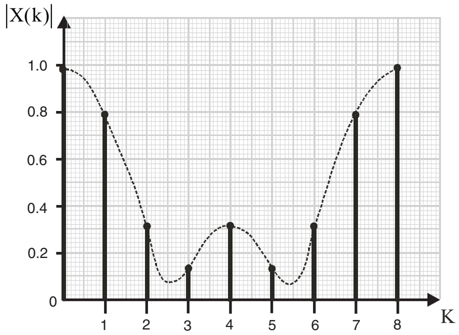


Fig 2: *Magnitude spectrum of X(k) for N=8.*

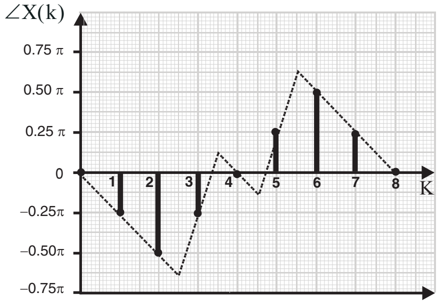


Fig 5: *Phase spectrum of X(k) for N=8.*

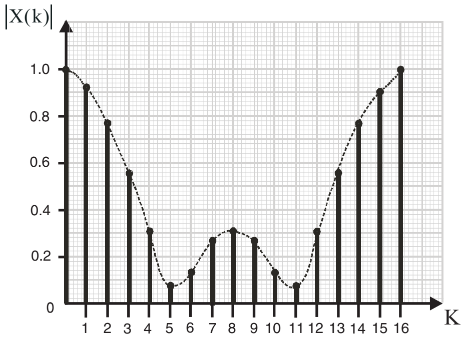


Fig 3: *Magnitude spectrum of X(k) for N=16*

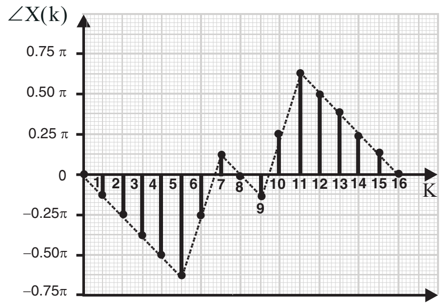


Fig 6: *Phase spectrum of X(k) for N=16.*

Example 1.7

Compute the DFT of the sequence, x(n) = {0, 1, 2, 3}. Sketch the magnitude and phase spectrum.

Solution

By the definition of DFT, the 4-point DFT is given by,

Here, x(n) is 4-point sequence,
∴ compute 4-point DFT

$$X(k) = \sum_{n=0}^{4-1} x(n)\, e^{\frac{-j2\pi kn}{4}} = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\pi kn}{2}}$$

x(0) = 0, x(1) = 1, x(2) = 2, x(3) = 3

$$= x(0)\, e^{0} + x(1)\, e^{\frac{-j\pi k}{2}} + x(2)\, e^{-j\pi k} + x(3)\, e^{\frac{-j3\pi k}{2}}$$

$$= 0 + e^{\frac{-j\pi k}{2}} + 2\, e^{-j\pi k} + 3\, e^{\frac{-3j\pi k}{3}}$$

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$= \left(\cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right) + 2(\cos\pi k - j\sin\pi k) + 3\left(\cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}\right)$$

$$\text{When } k=0\ ;\ X(0) = (\cos 0 - j\sin 0) + 2(\cos 0 - j\sin 0) + 3\,(\cos 0 + j\sin 0)$$

$$= (1 - j0) + 2(1 - j0) + 3(1 - j0)$$

$$= 6 = 6\angle 0$$

$$\text{When } k=1\ ;\ X(1) = \left(\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right) + 2(\cos\pi - j\sin\pi) + 3\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right)$$

$$= (0 - j1) + 2(-1 - j0) + 3(0 + j1) = -2 + j2$$

$$= 2.8\angle 135^\circ = 2.8\angle 135^\circ \times \frac{\pi}{180^\circ} = 2.8\angle 0.75\pi$$

$$\text{When } k=2\ ;\ X(2) = \left(\cos\frac{2\pi}{2} - j\sin\frac{2\pi}{2}\right) + 2(\cos 2\pi - j\sin 2\pi) + 3\left(\cos\frac{6\pi}{2} - j\sin\frac{6\pi}{2}\right)$$

$$= (-1 - j0) + 2(1 - j0) + 3(-1 + j0) = -2$$

$$= 2\angle 180^\circ = 2\angle 180^\circ \times \frac{\pi}{180^\circ} = 2\angle\pi$$

$$\text{when } k=3\ ;\ X(3) = \left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) - 2(\cos 3\pi - j\sin 3\pi) + 3\left(\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right)$$

$$= (0 + j1) + 2(-1 - j0) + 3(0 - j1) = -2 - j2$$

$$= 2.8\angle -135^\circ = 2.8\angle -135^\circ \times \frac{\pi}{180^\circ} = 2.8\angle -0.75\pi$$

$\therefore$ $X(k) = \{\,6\angle 0,\ \ 2.8\angle 0.75\pi,\ \ 2\angle\pi,\ \ 2.8\angle{-0.75\pi}\}$

Magnitude Spectrum, $|X(k)| = \{\,6,\ \ 2.8,\ \ 2,\ \ 2.8\,\}$

Phase Spectrum, $\angle X(k) = \{\,0,\ \ 0.75\pi,\ \ \pi,\ \ -0.75\pi\,\}$

Phase angles are in radians.

Note: When $k = 4$, $X(4) = \left(\cos\frac{4\pi}{2} - j\sin\frac{4\pi}{2}\right) - 2(\cos 4\pi - j\sin 4\pi) + 3\left(\cos\frac{12\pi}{2} - j\sin\frac{12\pi}{2}\right)$

$$= (1 - j0) + 2(1 - j0) + 3(1 - j0) = 6 = 6\angle 0$$

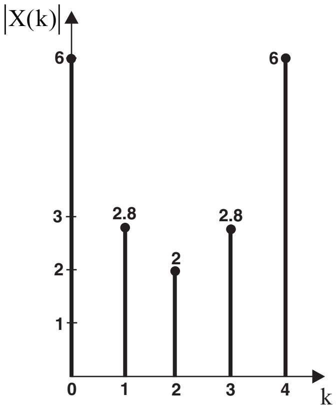


Fig 1: *Magnitude spectrum.*

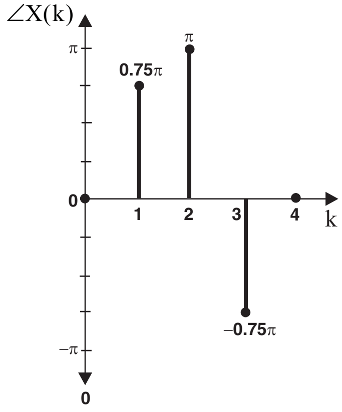


Fig 2: *Phase spectrum.*

Example 1.8

Compute the DFT of the sequence, x(n) = {0, 1, 2, 1}. Sketch the magnitude and phase spectrum.

Solution

The given signal x(n) is 4-point signal and so, let us compute 4-point DFT.

By the definition of DFT, the 4-point DFT is given by,

$$X(k) = \sum_{n=0}^{4-1} x(n)\, e^{\frac{-j2\pi kn}{4}} = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\pi kn}{2}}$$

$$= x(0)\, e^{0} + x(1)\, e^{\frac{-j\pi k}{2}} + x(2)\, e^{-j\pi k} + x(3)\, e^{\frac{-j3\pi k}{2}} = 0 + e^{\frac{-j\pi k}{2}} + 2\, e^{-j\pi k} + e^{\frac{-j3\pi k}{2}}$$

$$= \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2} + 2(\cos\pi k - j\sin\pi k) + \cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}$$

$$= \left(\cos\frac{\pi k}{2} + 2\cos\pi k + \cos\frac{3\pi k}{2}\right) - j\left(\sin\frac{\pi k}{2} + \sin\frac{3\pi k}{2}\right)$$

When $k = 0$; $X(0) = (\cos 0 + 2\cos 0 + \cos 0) - j(\sin 0 + \sin 0)$

$$= (1 + 2 + 1) - j(0 + 0) = 4 = 4\angle 0$$

When $k = 1$; $X(1) = \left(\cos\frac{\pi}{2} + 2\cos\pi + \cos\frac{3\pi}{2}\right) - j\left(\sin\frac{\pi}{2} + \sin\frac{3\pi}{2}\right)$

$$= (0 - 2 + 0) - j(1 - 1) = -2 = 2\angle 180^\circ = 2\angle -\pi$$

When $k = 2$; $X(2) = (\cos\pi + 2\cos 2\pi + \cos 3\pi) - j(\sin\pi + \sin 3\pi)$

$$= (-1 + 2 - 1) - j(0 + 0) = 0\angle 0$$

When $k = 3$; $X(3) = \left(\cos\frac{3\pi}{2} + 2\cos 3\pi + \cos\frac{9\pi}{2}\right) - j\left(\sin\frac{3\pi}{2} + \sin\frac{9\pi}{2}\right)$

$$= (0 - 2 + 0) - j(-1 + 1) = -2 = 2\angle 180^\circ = 2\angle\pi$$

$\therefore X(k) = \{ 4\angle 0,\ 2\angle -\pi,\ 0\angle 0,\ 2\angle\pi \}$

Magnitude Spectrum, $|X(k)| = \{ 4,\ 2,\ 0,\ 2 \}$

Phase Spectrum, $\angle X(k) = \{ 0,\ -\pi,\ 0,\ \pi \}$

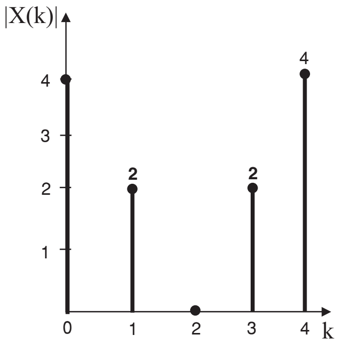


Fig 1: *Magnitude spectrum.*

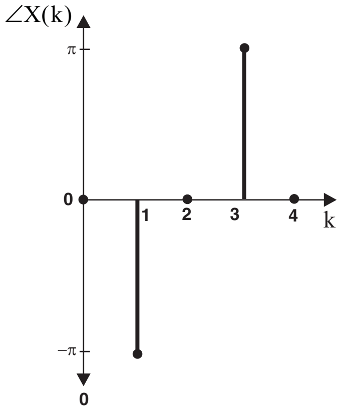


Fig 2: *Phase spectrum.*

Note: When $k = 4$, $X(4) = (\cos 2\pi + 2\cos 4\pi + \cos 6\pi) - j(\sin 2\pi + \sin 6\pi)$

$$= (1 + 2 + 1) - j(0 + 0) = 4 = 4\angle 0$$

1.6 Properties of DFT

1.6.1 Periodicity

If a sequence x(n) is periodic with periodicity of N samples, then N-point DFT, X(k) is also periodic with a periodicity of N samples.

Hence, if x(n) and X(k) are N point DFT pair then,

$$x(n+N) = x(n) \quad ; \quad \text{for all n}$$

$$X(k+N) = X(k) \quad ; \quad \text{for all k}$$

Proof:

By definition of DFT, the (k + N)th coefficient of X(k) is given by,

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n(k+N)}{N}} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}\, e^{\frac{-j2\pi nN}{N}}$$

Using definition of DFT.

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}\, e^{-j2\pi n} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$$

for integer n, $e^{-j2\pi n} = 1$.

$$= X(k)$$

1.6.2 Symmetry

Let x(n) be a complex valued discrete time signal of length, N. Now, x(n) can be expressed as sum of two sequences as shown in equation (1.44).

$$x(n) = x_r(n) + jx_i(n) \quad ; \quad \text{for } n = 0, 1, 2, \ldots.., N-1 \qquad .....(1.44)$$

where, $x_r(n)$ = Sequence consisting of real parts of x(n)

$x_i(n)$ = Sequence consisting of imaginary parts of x(n)

Let, $X(k) = \mathcal{DFT}\{x(n)\}$

Now by definition of N-point DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi nk}{N}} \quad ; \quad \text{for } k = 0, 1, 2, \ldots\ldots, N-1$$

Using equation (1.44).

$$= \sum_{n=0}^{N-1} \left(x_r(n) + jx_i(n)\right) e^{-\frac{j2\pi nk}{N}}$$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$$= \sum_{n=0}^{N-1} \left(x_r(n) + jx_i(n)\right)\left(\cos\frac{2\pi nk}{N} - j\sin\frac{2\pi nk}{N}\right)$$

$$= \sum_{n=0}^{N-1} \left(x_r(n)\cos\frac{2\pi nk}{N} - jx_r(n)\sin\frac{2\pi nk}{N} + jx_i(n)\cos\frac{2\pi nk}{N} + x_i(n)\sin\frac{2\pi nk}{N}\right)$$

$$= \sum_{n=0}^{N-1} \left(x_r(n)\cos\frac{2\pi nk}{N} + x_i(n)\sin\frac{2\pi nk}{N}\right) - j\sum_{n=0}^{N-1} \left(x_r(n)\sin\frac{2\pi nk}{N} - x_i(n)\cos\frac{2\pi nk}{N}\right) \qquad .....(1.45)$$

$$= X_r(k) + jX_i(k) \quad ; \quad \text{for } k = 0, 1, 2, \ldots\ldots, N-1$$

where, $X_r(k)$ = Sequence consisting of real parts of X(k)

$X_i(k)$ = Sequence consisting of imaginary parts of X(k)

Here, $$X_r(k) = \sum_{n=0}^{N-1}\left(x_r(n)\cos\frac{2\pi nk}{N} + x_i(n)\sin\frac{2\pi nk}{N}\right) \qquad .....(1.46)$$

$$X_i(k) = -\sum_{n=0}^{N-1}\left(x_r(n)\sin\frac{2\pi nk}{N} - x_i(n)\cos\frac{2\pi nk}{N}\right) \qquad .....(1.47)$$

Case (i): ***$x(n)$ is real-valued signal ($x(n) = x_r(n)$)***

Now, $x_i(n) = 0$

On substituting $x_i(n) = 0$, in equation (1.45) we get,

$$X(k) = \sum_{n=0}^{N-1} x_r(n)\cos\frac{2\pi nk}{N} - j\sum_{n=0}^{N-1} x_r(n)\sin\frac{2\pi nk}{N}$$

$$\boxed{x_r(n) = x(n)}$$

$$= \sum_{n=0}^{N-1} x(n)\cos\frac{2\pi nk}{N} - j\sum_{n=0}^{N-1} x(n)\sin\frac{2\pi nk}{N} \qquad .....(1.48)$$

$$= \sum_{n=0}^{N-1} x(n)\left(\cos\frac{2\pi nk}{N} - j\sin\frac{2\pi nk}{N}\right)$$

$$\boxed{\cos\theta - j\sin\theta = e^{-j\theta}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi nk}{N}} \qquad .....(1.49)$$

On substituting $k = N-k$ in equation (1.49) we get,

$$X(n-k) = \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi n(N-k)}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi nN}{N}}\, e^{\frac{j2\pi nk}{N}}$$

$$\boxed{\text{Since, n is integer, } e^{\frac{j2\pi nN}{N}} = e^{-j2\pi n} = 1.}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi nk}{N}}$$

$$= \left(\sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi nk}{N}}\right)^{*}$$

$$= X^{*}(k)$$

$$\boxed{\text{Using equation (1.49).}}$$

Also, $X(n-k) = X(-k)$

$\therefore X(-k) = X^{*}(k)$

From equation (1.49),

$$\boxed{e^{-\frac{j2\pi nN}{N}} \text{ is a phase factor and its magnitude contribution is 1.}}$$

$$\left|X(k)\right| = \left|\sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi nk}{N}}\right|$$

$$= \left|\sum_{n=0}^{N-1} x(n)\right|\left|e^{-\frac{j2\pi nk}{N}}\right| = \left|\sum_{n=0}^{N-1} x(n)\right| \times 1$$

$$\boxed{\left|e^{\pm\frac{j2\pi nN}{N}}\right| = 1.}$$

$$= \left|\sum_{n=0}^{N-1} x(n)\right|$$

Similarly, $\left|X(N-k)\right| = \left|\sum_{n=0}^{N-1} x(n)\right|$ and so $\left|X(N-k)\right| = \left|X(k)\right|$

Since x(n) is real, from equation (1.49) we can say that the phase is contributed only by the term, $e^{-\frac{j2\pi nk}{N}}$.

$$\therefore\ \angle X(k) = \angle e^{-\frac{j2\pi nk}{N}} = -\frac{2\pi nk}{N}$$

$$\therefore\ \angle X(-k) = \frac{2\pi nk}{N} = -\angle X(k)$$

Also, $\therefore\ \angle X^*(k) = \frac{2\pi nk}{N} = -\angle X(k)$

Case (ii):* *x(n) is real and even signal

Since x(n) is real and even, it will satisfy the condition,

$x(n) = x(N-n)$; for $n = 0, 1, 2, ..., N-1$

Since x(n) is real, $x_i(n) = 0$.

On substituting $x_i(n) = 0$, in equation (1.45) we get,

$$X(k) = \sum_{n=0}^{N-1} x_r(n)\cos\frac{2\pi nk}{N} - j\sum_{n=0}^{N-1} x_r(n)\sin\frac{2\pi nk}{N}$$

In the above equation,

$x_r(n) = x(n)$ - even signal

$\sin\frac{2\pi nk}{N}$ - odd signal

$\cos\frac{2\pi nk}{N}$ - even signal

$\therefore\ x_r(n)\cos\frac{2\pi nk}{N}$ - even signal

$\therefore\ x_r(n)\sin\frac{2\pi nk}{N}$ - odd signal

Product of even and odd signal is an odd signal.

Product of even signals is an even signal.

$$\therefore\ \sum_{n=0}^{N-1} x_r(n)\sin\frac{2\pi nk}{N} = 0$$

Sum of samples of one period of odd signal is zero.

$$\therefore\ X(k) = \sum_{n=0}^{N-1} x(n)\cos\frac{2\pi nk}{N} \quad ; \quad \text{for } k = 0, 1, 2, ..., N-1$$

From the above discussion we can say that, if x(n) is real and even then X(k) is purely real and even.

$\therefore\ X(k) = X_r(k)$ and $X_i(k) = 0$

Case (iii):* *x(n) is real and odd signal

Since x(n) is real and odd, it will satisfy the condition,

$x(n) = -x(N-n)$; for $n = 0, 1, 2, ..., N-1$

Since x(n) is real, $x_i(n) = 0$.

On substituting $x_i(n) = 0$, in equation (1.45) we get,

$$X(k) = \sum_{n=0}^{N-1} x_r(n)\cos\frac{2\pi nk}{N} - j\sum_{n=0}^{N-1} x_r(n)\sin\frac{2\pi nk}{N}$$

In the above equation,

$x_r(n) = x(n)$ - odd signal

$\cos\frac{2\pi nk}{N}$ - even signal

$\sin\frac{2\pi nk}{N}$ - odd signal

$x_r(n)\sin\frac{2\pi nk}{N}$ - even signal

$\therefore\ x_r(n)\cos\frac{2\pi nk}{N}$ - odd signal

Product of odd and even signal is an odd signal.

Product of odd signals is an even signal.

$$\therefore\ \sum_{n=0}^{N-1} x_r(n)\cos\frac{2\pi nk}{N} = 0$$

Sum of samples of one period of odd signal is zero.

$$\therefore\ X(k) = -j\sum_{n=0}^{N-1} x(n)\sin\frac{2\pi nk}{N} \quad ; \quad \text{for } k = 0, 1, 2, ..., N-1$$

From the above discussion we can say that, if x(n) is real and odd then X(k) is purely imaginary and odd.

$\therefore\ X(k) = -jX_i(k)$ and $X_r(k) = 0$

Case (iv): ***x(n) is purely imaginary signal ($x(n) = jx_i(n)$)***

Now, $x_r(n) = 0$

On substituting $x_r(n) = 0$, in equation (1.45) we get,

$$X(k) = \sum_{n=0}^{N-1} x_i(n)\sin\frac{2\pi nk}{N} + j\sum_{n=0}^{N-1} x_i(n)\cos\frac{2\pi nk}{N} \quad ; \quad \text{for } k = 0, 1, 2, ..., N-1$$

$$= X_r(k) + jX_i(k) \qquad .....(1.50)$$

where, $X_r(k)$ = Real part of $X(k) = \sum_{n=0}^{N-1} x_i(n)\sin\frac{2\pi nk}{N}$; for $k = 0, 1, 2, ..., N-1$

$X_i(k)$ = Imaginary part of $X(k) = \sum_{n=0}^{N-1} x_i(n)\cos\frac{2\pi nk}{N}$; for $k = 0, 1, 2, ..., N-1$

If x(n) is purely imaginary and even signal then,

$x_i(n)$ - even signal

$\sin\frac{2\pi nk}{N}$ - odd signal

$\cos\frac{2\pi nk}{N}$ - even signal

$\therefore\ x_i(n)\cos\frac{2\pi nk}{N}$ - even signal

$\therefore\ x_i(n)\sin\frac{2\pi nk}{N}$ - odd signal

Product of even and odd signal is an odd signal.

Product of even signals is an even signal.

$$\therefore\ \sum_{n=0}^{N-1} x_i(n)\sin\frac{2\pi nk}{N} = 0$$

Sum of samples of one period of odd signal is zero.

$$\therefore\ X(k) = j\sum_{n=0}^{N-1} x_i(n)\cos\frac{2\pi nk}{N} = jX_i(k)$$

From the above discussion we can say that, if x(n) is imaginary and even then X(k) is purely imaginary and even.

If x(n) is purely imaginary and odd signal then,

$x_i(n)$ - odd signal

$\cos\frac{2\pi nk}{N}$ - even signal

$\sin\frac{2\pi nk}{N}$ - odd signal

$\therefore\ x_i(n)\sin\frac{2\pi nk}{N}$ - even signal

$\therefore\ x_i(n)\cos\frac{2\pi nk}{N}$ - odd signal

Product of odd and even signal is an odd signal.

Product of odd signals is an even signal.

$$\therefore \sum_{n=0}^{N-1} x_i(n) \cos \frac{2\pi nk}{N} = 0$$

Sum of samples of one period of odd signal is zero.

$$\therefore X(k) = \sum_{n=0}^{N-1} x_i(n) \sin \frac{2\pi nk}{N} = X_r(k)$$

From the above discussion we can say that, if x(n) is imaginary and odd then X(k) is purely real and odd.

Note: *The symmetry properties of DFT are summarized in Table 1.4.*

1.6.3 Circular Convolution

The convolution property of DFT says that the DFT of circular convolution of two sequences is equivalent to the product of their individual DFTs.

Let, $\mathcal{DFT}\{x_1(n)\} = X_1(k)$ and $\mathcal{DFT}\{x_2(n)\} = X_2(k)$, then by convolution property,

$$\mathcal{DFT}\{x_1(n) \circledast x_2(n)\} = X_1(k)\, X_2(k)$$

Proof:

Let, $x_1(n)$ and $x_2(n)$ be N-point sequences. Now by definition of DFT,

Let, n = m

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{m=0}^{N-1} x_1(m)\, e^{\frac{-j2\pi km}{N}} \quad ; \quad \text{for } k = 0, 1, 2, \ldots\ldots, N-1 \qquad \text{.....(1.51)}$$

Let, n = p

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{p=0}^{N-1} x_2(p)\, e^{\frac{-j2\pi kp}{N}} \quad ; \quad \text{for } k = 0, 1, 2, \ldots\ldots, N-1 \qquad \text{.....(1.52)}$$

Consider the product $X_1(k)\, X_2(k)$. The inverse DFT of the product is given by,

$$\mathcal{DFT}^{-1}\{X_1(k)\, X_2(k)\} = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k)\, X_2(k) e^{\frac{j2\pi kn}{N}}$$

Using the equations (1.51) and (1.52).

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{m=0}^{N-1} x_1(m)\, e^{\frac{-j2\pi km}{N}} \right] \left[\sum_{p=0}^{N-1} x_2(p)\, e^{\frac{-j2\pi kp}{N}} \right] e^{\frac{j2\pi kn}{N}}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \sum_{p=0}^{N-1} x_2(p) \sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m-p)}{N}} \qquad \text{.....(1.53)}$$

Consider the summation $\sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m-p)}{N}}$ in equation (1.53).

Let, n – m – p = qN, where q is an integer.

Since q is an integer, $e^{j2\pi q} = 1$.

$$\therefore \sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m-p)}{N}} = \sum_{k=0}^{N-1} e^{\frac{j2\pi kqN}{N}} = \sum_{k=0}^{N-1} \left(e^{j2\pi q}\right)^k = \sum_{k=0}^{N-1} 1^k = N \qquad \text{.....(1.54)}$$

Consider the summation $\sum_{p=0}^{N-1} x_2(p)$ in equation (1.53).

Since, n – m – p = qN, p = n – m – qN

$$\therefore \sum_{p=0}^{N-1} x_2(p) = \sum_{m=0}^{N-1} x_2(n-m-qN) = \sum_{m=0}^{N-1} x_2(n-m, \mathrm{mod}\, N) = \sum_{m=0}^{N-1} x_2((n-m))_N \qquad \text{.....(1.55)}$$

Using the equations (1.54) and (1.55), equation (1.53) can be written as shown below:

$$\mathcal{DFT}^{-1}\{X_1(k)\,X_2(k)\} = \frac{1}{N}\sum_{m=0}^{N-1} x_1(m)\sum_{m=0}^{N-1} x_2((n-m))_N\,N = \sum_{m=0}^{N-1} x_1(m)\,x_2((n-m))_N$$

$$= x_1(n) \circledast x_2(n)$$

$$\therefore\ X_1(k)\,X_2(k) = \mathcal{DFT}\{x_1(n) \circledast x_2(n)\}$$

Using definition of circular convolution.

1.6.4 Linearity

The linearity property of DFT states that the DFT of a linear weighted combination of two or more signals is equal to the similar linear weighted combination of the DFT of individual signals.

Let, $\mathcal{DFT}\{x_1(n)\} = X_1(k)$ and $\mathcal{DFT}\{x_2(n)\} = X_2(k)$ then by linearity property,

$\mathcal{DFT}\{a_1\,x_1(n) + a_2\,x_2(n)\} = a_1\,X_1(k) + a_2\,X_2(k)$, where a_1 and a_2 are constants.

Proof:

By the definition of discrete Fourier transform,

$$X_1(k) = \mathcal{DFT}\{x_1(n)\} = \sum_{n=0}^{N-1} x_1(n)\,e^{\frac{-j2\pi kn}{N}} \qquad .....(1.56)$$

$$X_2(k) = \mathcal{DFT}\{x_2(n)\} = \sum_{n=0}^{N-1} x_2(n)\,e^{\frac{-j2\pi kn}{N}} \qquad .....(1.57)$$

$$\mathcal{DFT}\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=0}^{N-1}[a_1 x_1(n) + a_2 x_2(n)]\,e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1}\left[a_1 x_1(n) e^{\frac{-j2\pi kn}{N}} + a_2 x_2(n) e^{\frac{-j2\pi kn}{N}}\right]$$

$$= a_1\sum_{n=0}^{N-1} x_1(n) e^{\frac{-j2\pi kn}{N}} + a_2\sum_{n=0}^{N-1} x_2(n) e^{\frac{-j2\pi kn}{N}}$$

$$= a_1 X_1(k) + a_2 X_2(k)$$

Using equations (1.56) and (1.57).

1.6.5 Circular Time Shift

The circular time shift property of DFT says that if a discrete time signal is circularly shifted in time by m units, then its DFT is multiplied by $e^{\frac{-j2\pi km}{N}}$.

i.e., if $\mathcal{DFT}\{x(n)\} = X(k)$, then $\mathcal{DFT}\{x((n-m))_N\} = X(k)\,e^{\frac{-j2\pi km}{N}}$

Proof:

$$\mathcal{DFT}\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N\,e^{\frac{-j2\pi kn}{N}} = \sum_{p=0}^{N-1} x(p)\,e^{\frac{-j2\pi k(p+m)}{N}}$$

$$= \sum_{p=0}^{N-1} x(p)\,e^{\frac{-j2\pi kp}{N}}\;e^{\frac{-j2\pi km}{N}}$$

Let $p = n - m,\ \therefore\ n = p + m$

$$= \left[\sum_{p=0}^{N-1} x(p)\,e^{\frac{-j2\pi kp}{N}}\right] e^{\frac{-j2\pi km}{N}}$$

$$= X(k)\,e^{\frac{-j2\pi km}{N}}$$

Using definition of DFT.

1.6.6 Time Reversal

The time reversal property of DFT says that reversing the N-point sequence in time is equivalent to reversing the DFT sequence.

i.e., if, $\mathcal{DFT}\{x(n)\} = X(k)$, then $\mathcal{DFT}\{x(N-n)\} = X(N-k)$.

Proof:

Let m = N – n ∴ n = N – m

$$\mathcal{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi k(N-m)}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi kN}{N}}\, e^{\frac{j2\pi km}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}}\, e^{-j2\pi k}$$

Since k is an integer, $e^{-j2\pi k} = 1$.

Since m is an integer, $e^{-j2\pi m} = 1$.

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}}\, e^{-j2\pi m}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}}\, e^{\frac{j2\pi mN}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi m(N-k)}{N}}$$

$$= X(N-k)$$

Using definition of DFT.

1.6.7 Conjugation

Let, x(n) be a complex N-point discrete sequence and x*(n) be its conjugate sequence.

Now, if $\mathcal{DFT}\{x(n)\} = X(k)$, then $\mathcal{DFT}\{x^*(n)\} = X^*(N-k)$.

Proof:

$e^{-j2\pi n} = 1$

$$\mathcal{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\, e^{\frac{-j2\pi kn}{N}} = \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}}\, e^{-j2\pi n}\right]^* = \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}}\, e^{\frac{-j2\pi nN}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n(N-k)}{N}}\right]^* = [X(N-k)]^* = X^*(N-k)$$

Using definition of DFT.

1.6.8 Circular Frequency Shift

The circular frequency shift property of DFT says that if a discrete time signal is multiplied by $e^{j\frac{2\pi mn}{N}}$, then its DFT is circularly shifted by m units.

i.e., if, $\mathcal{DFT}\{x(n)\} = X(k)$, then $\mathcal{DFT}\{x(n)\, e^{j\frac{2\pi mn}{N}}\} = X((k-m))_N$.

Proof:

$$\mathcal{DFT}\{x(n)\, e^{\frac{j2\pi mn}{N}}\} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi mn}{N}}\, e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi (k-m)n}{N}} = X((k-m))_N$$

Using definition of DFT.

1.6.9 Multiplication

The multiplication property of DFT says that the DFT of product of two discrete time sequences is equivalent to the circular convolution of the DFTs of the individual sequences scaled by a factor 1/N.

i.e., if, $\mathcal{DFT}\{x(n)\} = X(k)$, then $\mathcal{DFT}\{\, x_1(n)\, x_2(n)\,\} = \frac{1}{N}[X_1(k) \circledast X_2(k)]$

Proof:

By definition of inverse DFT,

Let, k = m

$$x_1(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, e^{\frac{j2\pi kn}{N}} = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\, e^{\frac{j2\pi mn}{N}} \qquad .....(1.58)$$

By definition of DFT,

$$\mathcal{DFT}\{x_1(n)x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)x_2(n)e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1(m)e^{\frac{j2\pi mn}{N}}\right] x_2(n)e^{\frac{-j2\pi kn}{N}}$$

Using equation (1.58).

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{-j2\pi kn}{N}}\; e^{\frac{j2\pi mn}{N}}\right]$$

Rearranging the order of summation.

Using definition of DFT.

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)e^{\frac{-j2\pi(k-m)n}{N}}\right] = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)X_2((k-m))_N$$

$$= \frac{1}{N}[X_1(k) \circledast X_2(k)]$$

Using definition of circular convolution.

Note: *The circular convolution of two N-point sequences $x_1(n)$ and $x_2(n)$ is defined as,*

$$x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m)x_2((n-m))_N$$

Refer equation (1.17).

1.6.10 Circular Correlation

The circular correlation of two sequences x(n) and y(n) is defined as,

$$\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n)\, y^*((n-m))_N$$

Let $\mathcal{DFT}\{x(n)\} = X(k)$ and $\mathcal{DFT}\{y(n)\} = Y(k)$, then by correlation property,

$$\mathcal{DFT}\{\bar{r}_{xy}(m)\} = \mathcal{DFT}\left\{\sum_{n=0}^{N-1} x(n)\, y^*((n-m))_N\right\} = X(k)Y^*(k)$$

Proof:

Let, x(n) and y(n) be N-point sequences. Now by definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \text{ for } k = 0, 1, 2, ..., N-1 \qquad .....(1.59)$$

Let, n = p

$$Y(k) = \sum_{n=0}^{N-1} y(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{p=0}^{N-1} y(p)\, e^{\frac{-j2\pi kp}{N}} \quad ; \text{ for } k = 0,1,2, ..., N-1 \qquad .....(1.60)$$

Consider the product X(k) Y*(k). The inverse DFT of the product is given by,

$$\mathcal{DFT}^{-1}\{X(k)\,Y^*(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,Y^*(k)\,e^{\frac{j2\pi kn}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,Y^*(k)\,e^{\frac{j2\pi km}{N}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x(n)\;e^{\frac{-j2\pi kn}{N}}\right]\left[\sum_{p=0}^{N-1} y(p)\;e^{\frac{-j2\pi kp}{N}}\right]^* e^{\frac{j2\pi km}{N}}$$

Using the equations (1.59) and (1.60).

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\sum_{p=0}^{N-1} y^*(p)\sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n+p)}{N}} \qquad .....(1.61)$$

Consider the summation $\sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n+p)}{N}}$ in equation (1.61).

Let, m – n + p = qN, where q is an integer.

Since q is an integer, $e^{j2\pi q} = 1$.

$$\therefore \sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n+p)}{N}} = \sum_{k=0}^{N-1} e^{\frac{j2\pi kqN}{N}} = \sum_{k=0}^{N-1} (e^{j2\pi q})^k = \sum_{k=0}^{N-1} 1^k = N \qquad .....(1.62)$$

Consider the summation $\sum_{p=0}^{N-1} y^*(p)$ in equation (1.61).

Since, m – n + p = qN, p = n – m + qN

$$\therefore \sum_{p=0}^{N-1} y^*(p) = \sum_{n=0}^{N-1} y^*(n-m+qN) = \sum_{n=0}^{N-1} y^*(n-m, \mathrm{mod}\,N) = \sum_{n=0}^{N-1} y^*((n-m))_N \qquad .....(1.63)$$

Using the equations (1.62) and (1.63), equation (1.61) can be written as shown below:

$$\mathcal{DFT}^{-1}\{X(k)\,Y^*(k)\} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\sum_{n=0}^{N-1} y^*((n-m))_N N$$

$$= \sum_{n=0}^{N-1} x(n)\,y^*((n-m))_N = \overline{r_{xy}}(m)$$

$$\therefore\; X(k)\,Y^*(k) = \mathcal{DFT}\left\{\overline{r_{xy}}(m)\right\}$$

Using definition of circular convolution.

1.6.11 Parseval's Relation

Let, $\mathcal{DFT}\{x_1(n)\} = X_1(k)$ and $\mathcal{DFT}\{x_2(n)\} = X_2(k)$ then by Parseval's relation,

$$\sum_{n=0}^{N-1} x_1(n)\,x_2^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\,X_2^*(k)$$

Proof:

Let, $x_1(n)$ and $x_2(n)$ be N- point sequences.

Now by definition of DFT, $X_1(k) = \sum_{n=0}^{N-1} x_1(n)\,e^{\frac{-j2\pi kn}{N}}$ .....(1.64)

Now by definition of inverse DFT, $x_2(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_2(k)\,e^{\frac{j2\pi kn}{N}}$ .....(1.65)

Consider the right-hand side term of Parseval's relation,

$$\frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\,X_2^*(k) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)\,e^{\frac{-j2\pi kn}{N}}\right]X_2^*(k)$$

Using equation (1.64).

$$= \sum_{n=0}^{N-1} x_1(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} X_2^*(k)\,e^{\frac{-j2\pi kn}{N}}\right] = \sum_{n=0}^{N-1} x_1(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} X_2(k)\,e^{\frac{j2\pi kn}{N}}\right]^*$$

$$= \sum_{n=0}^{N-1} x_1(n)\,x_2^*(n)$$

Using equation (1.65).

Table 1.4: Properties of Discrete Fourier Transform (DFT)

Property	Discrete Time Signal	Discrete Fourier Transform
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Periodicity	$x(n + N) = x(n)$	$X(k + N) = X(k)$
Circular time shift	$x((n - m))_N$	$X(k)\ e^{\frac{-j2\pi km}{N}}$
Time reversal	$x(N - n)$	$X(N - k)$
Conjugation	$x^*(n)$	$X^*(N - k)$
Circular frequency shift	$x(n)\ e^{\frac{j2\pi mn}{N}}$	$X((k - m))_N$
Multiplication	$x_1(n)\ x_2(n)$	$\frac{1}{N}[X_1(k) \circledast X_2(k)]$
Circular convolution	$x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m)\,x_2((n-m))_N$	$X_1(k)\ X_2(k)$
Circular correlation	$\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n)\,y^*((n-m))_N$	$X(k)\ Y^*(k)$
Symmetry of real signals	x(n) is real	$X(k) = X^*(N-k)$ $X_r(k) = X_r(N-k)$ $X_i(k) = -X_i(N-k)$ $\lvert X(k)\rvert = \lvert X(N-k)\rvert$ $\angle X(k) = -\angle X(N-k)$
Symmetry of real and even signal	x(n) is real and even $x(n) = x(N - n)$	$X(k) = X_r(k)$ and $X_i(k) = 0$
Symmetry of real and odd signal	x(n) is real and odd $x(n) = -x(N - n)$	$X(k) = j\ X_i(k)$ and $X_r(k) = 0$
Parseval's relation	$\sum_{n=0}^{N-1} x_1(n)\,x_2^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\,X_2^*(k)$

1.7 Linear Filtering using DFT

The response of an LTI system is given by linear convolution of input and impulse response of the system. A filter is basically an LTI system and so the response of a filter is given by the linear convolution of input and impulse response of the filter.

Let, x(n) = Input to filter

h(n) = Impulse response of the filter

y(n) = Output or response of the filter

Now, the response or output of the filter y(n) is given by linear convolution of x(n) and h(n) as shown below:

$$\text{Response, } y(n) = x(n) * h(n)$$

$$\text{where, } x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) \qquad .....(1.66)$$

The DFT supports only circular convolution and so the linear convolution of equation (1.66) has to be computed via circular convolution. If x(n) is N_1-point sequence and h(n) is N_2-point sequence, then linear convolution of x(n) and h(n) will generate y(n) of size $N_1 + N_2 - 1$.

Therefore, in order to perform linear convolution via circular convolution, the x(n) and h(n) should be converted to $N_1 + N_2 - 1$ point sequences by appending zeros. Now the circular convolution of $N_1 + N_2 - 1$ point sequences x(n) and h(n) will give the same result as that obtained by linear convolution.

Let, x(n) be N_1-point sequence and h(n) be N_2-point sequence.

Let us convert x(n) and h(n) to $N_1 + N_2 - 1$ point sequences.

Let, $Y(k) = N_1 + N_2 - 1$ point DFT of y(n)

$X(k) = N_1 + N_2 - 1$ point DFT of x(n)

$H(k) = N_1 + N_2 - 1$ point DFT of h(n)

Now, by circular convolution theorem of DFT,

$$\mathcal{DFT}\{x(n) \circledast h(n)\} = X(k)\, H(k)$$

On taking inverse DFT of the above equation we get,

$$x(n) \circledast h(n) = \mathcal{DFT}^{-1}\{X(k)\, H(k)\}$$

Since, $x(n) \circledast h(n) = y(n)$, the above equation can be written as,

$$\text{Response, } y(n) = \mathcal{DFT}^{-1}\{X(k)\, H(k)\} \qquad .....(1.67)$$

From equation (1.67), we can say that the filter output or response y(n) is given by the inverse DFT of the product of X(k) and H(k).

The following procedure can be followed to compute response of filter using DFT.

1. Let x(n) be N_1-point sequence and h(n) be N_2-point sequence and so the response y(n) of the filter is $N_1 + N_2 - 1$ point sequence. Let, $N_1 + N_2 - 1 = M$.

2. Compute M-point DFT of x(n) to get X(k).
3. Compute M-point DFT of h(n) to get H(k).
4. Determine the product of X(k) and H(k). Let, Y(k) = X(k) H(k).
5. Take M-point inverse DFT of Y(k) to get y(n) which is the response of the filter.

1.7.1 Filtering Long Data Sequences

In real time applications the input sequence of the filter may be very long or continuous stream of data. In such cases, the filter response is computed by dividing the input into small blocks of data and the response of each block is computed separately. Then overall response is obtained by combining the individual responses. This concept is discussed in sectioned convolution in Section 1.1.15.

In digital computers, the computation of sectioned convolution is performed via DFT. In sectioned convolution via DFT, the computation of convolution of each section is performed by DFT and IDFT as discussed in Section 1.7. In DFT techniques, the procedure of dividing the input sequence into smaller sequence, converting the sequences to size of output sequence and combining the result of output sequences are same as that of sectioned convolution discussed in Section 1.1.15. The only difference is the convolution of each section is performed via DFT and so in DFT techniques also we have two methods of sectioned convolution: overlap add method and overlap save method.

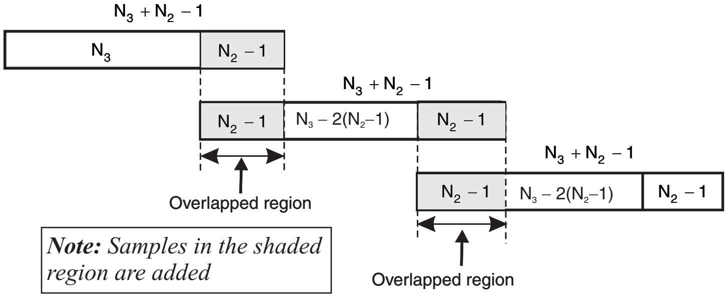
1.7.2 Overlap Add Method

In the ***overlap add method,*** the input sequence is divided into smaller sequences. Then linear convolution of each section of the input sequence and the filter impulse response sequence is performed. The overall filter output sequence is obtained by combining the output of the sectioned convolution.

Let, N_1 = Length of input sequence

N_2 = Length of filter impulse response sequence

Let the input sequence be divided into sections of size N_3 samples.



Note: *Samples in the shaded region are added*

Fig 1.32: *Overlapping of output sequence of sectioned convolution by overlap add method.*

The linear convolution of each section with input sequence will produce an output sequence of size $N_3 + N_2 - 1$ samples. Hence the sections of input sequence and filter impulse response sequence are converted to size of output sequence by appending zeros in order to compute linear convolution via circular convolution.

In this method the last $N_2 - 1$ samples of each output sequence overlap with the first $N_2 - 1$ samples of the next section [i.e., there will be a region of $N_2 - 1$ samples over which the output sequence of q^{th} convolution overlaps with the output sequence of $(q + 1)^{th}$ convolution]. While combining the output sequences of the various sectioned convolutions, the corresponding samples of overlapped regions are added and the samples of non-overlapped regions are retained as such.

1.7.3 Overlap Save Method

In the ***overlap save method,*** the sections of input sequences are converted to size of output sequence using samples of next or previous section in order to perform linear convolution via circular convolution. In this method, the input sequence is divided into smaller sequences. Each section of the input sequence and the impulse response sequence are converted to the size of the output sequence of sectioned convolution. The circular convolution of each section of the input sequence and the filter impulse response sequence is performed. The overall output sequence is obtained by combining the outputs of the sectioned convolution.

Let, N_1 = Length of input sequence

N_2 = Length of impulse response sequence

Let the input sequence be divided into sections of size N_3 samples.

The impulse response sequence is converted to the size of $N_3 + N_2 - 1$ samples, by appending with zeros. The conversion of each section of the input sequence to the size $N_3 + N_2 - 1$ samples can be performed by two different methods.

Method-1

In this method, the first $N_2 - 1$ samples of a section are appended as last $N_2 - 1$ samples of the previous section [i.e., the overlapping samples are placed at the beginning of the section]. The circular convolution of each section will produce an output sequence of size $N_3 + N_2 - 1$ samples. In this output, the first $N_2 - 1$ samples are discarded and the remaining samples of the output of sectioned convolutions are saved as the overall output sequence.

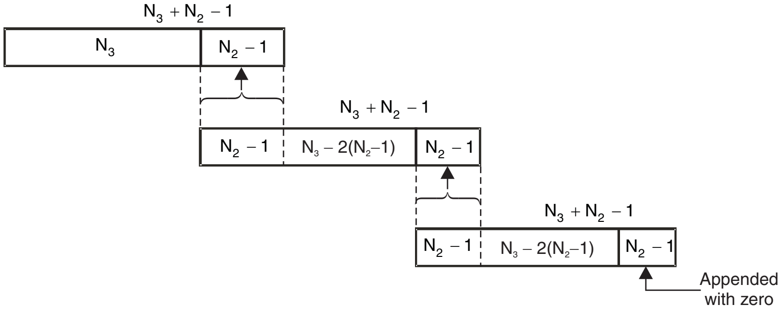


Fig 1.33: *Appending of sections of input sequence in method-1 of overlap save method.*

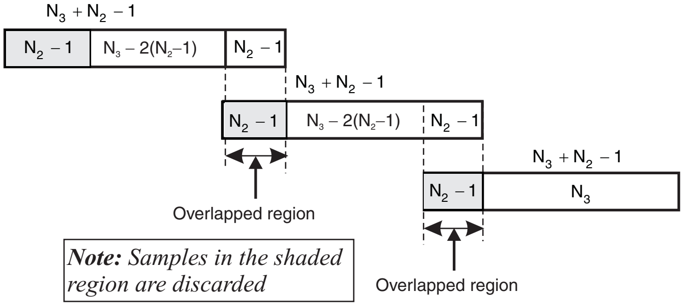


Fig 1.34: *Overlapping of output sequence of sectioned convolution by method-1 of overlap save method.*

Method-2

In this method, the last N_2–1 samples of a section are appended as last N_2–1 samples of the next section (i.e, the overlapping samples are placed at the end of the sections). The circular convolution of each section will produce an output sequence of size $N_3 + N_2 – 1$ samples. In this output, the last N_2–1 samples are discarded and the remaining samples of the output of sectioned convolutions are saved as the overall output sequence.

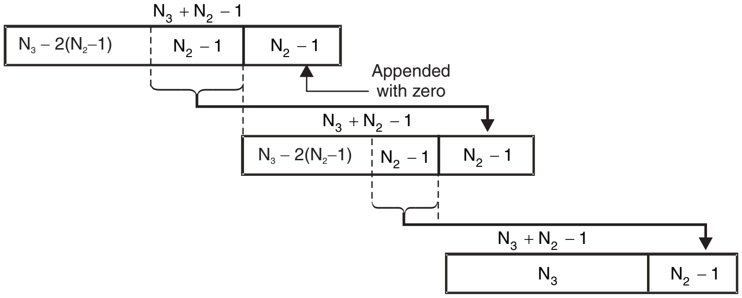


***Fig 1.35:** Appending of sections of input sequence in method-2 of overlap save method.*

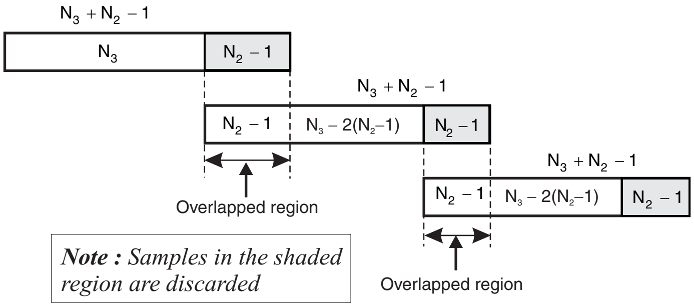


***Fig 1.36:** Overlapping of output sequence of sectioned convolution by method-2 of overlap save method.*

Example 1.9

Compute the circular convolution of the following two sequences using DFT.

$x_1(n) = \{ \underset{\uparrow}{0}, 1, 0, 1 \}$ and $x_2(n) = \{ \underset{\uparrow}{1}, 2, 1, 2 \}$

Solution

The 4-point DFT of $x_1(n)$ is,

$x_1(0) = 0$
$x_1(1) = 1$
$x_1(2) = 0$
$x_1(3) = 1$

$$\mathcal{DFT}\{x_1(n)\} = X_1(k) = \sum_{n=0}^{4-1} x_1(n)\, e^{\frac{-j2\pi kn}{4}} = \sum_{n=0}^{3} x_1(n)\, e^{\frac{-j\pi kn}{2}} \;;\; k = 0, 1, 2, 3$$

$$= x_1(0)\, e^0 + x_1(1)\, e^{\frac{-j\pi k}{2}} + x_1(2)\, e^{-j\pi k} + x_1(3)\, e^{\frac{-j3\pi k}{2}}$$

$$= 0 + e^{\frac{-j\pi k}{2}} + 0 + e^{\frac{-j3\pi k}{2}} = e^{\frac{-j\pi k}{2}} + e^{\frac{-j3\pi k}{2}}$$

When $k = 0$; $X_1(0) = e^0 + e^0 = 1 + 1 = 2$

When $k = 1$; $X_1(1) = e^{\frac{-j\pi}{2}} + e^{\frac{-j3\pi}{2}} = -j + j = 0$

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

When $k = 2$; $X_1(2) = e^{-j\pi} + e^{-j3\pi} = -1-1 = -2$

When $k = 3$; $X_1(3) = e^{\frac{-j3\pi}{2}} + e^{\frac{-j9\pi}{2}} = j - j = 0$

The 4-point DFT of $x_2(n)$ is,

$$\mathcal{DFT}\{x_2(n)\} = X_2(k) = \sum_{n=0}^{4-1} x_2(n)\, e^{\frac{-j2\pi kn}{4}} = \sum_{n=0}^{3} x_2(n)\, e^{\frac{-j\pi kn}{2}} \;;\; k = 0, 1, 2, 3$$

$$\therefore X_2(k) = x_2(0)\, e^0 + x_2(1)\, e^{\frac{-j\pi k}{2}} + x_2(2)\, e^{-j\pi k} + x_2(3)\, e^{\frac{-j3\pi k}{2}}$$

$$= 1 + 2\, e^{\frac{-j\pi k}{2}} + e^{-j\pi k} + 2\, e^{\frac{-j3\pi k}{2}}$$

When $k = 0$; $X_2(0) = 1 + 2e^0 + e^0 + 2e^0 = 1+2+1+2 = 6$

When $k = 1$; $X_2(1) = 1 + 2e^{\frac{-j\pi}{2}} + e^{-j\pi} + 2e^{\frac{-j3\pi}{2}} = 1 - 2j - 1 + 2j = 0$

When $k = 2$; $X_2(2) = 1 + 2e^{-j\pi} + e^{-j2\pi} + 2e^{-j3\pi} = 1 - 2 + 1 - 2 = -2$

When $k = 3$; $X_2(3) = 1 + 2e^{\frac{-j3\pi}{2}} + e^{-j3\pi} + 2e^{\frac{-j9\pi}{2}} = 1 + 2j - 1 - 2j = 0$

$$X_1(k) = \begin{cases} 2 \;;\; k=0 \\ 0 \;;\; k=1 \\ -2 \;;\; k=2 \\ 0 \;;\; k=3 \end{cases} \qquad X_2(k) = \begin{cases} 6 \;;\; k=0 \\ 0 \;;\; k=1 \\ -2 \;;\; k=2 \\ 0 \;;\; k=3 \end{cases}$$

Let $X_3(k)$ be the product of $X_1(k)$ and $X_2(k)$.

$$\therefore X_3(k) = X_1(k)\, X_2(k)$$

$\boxed{\begin{array}{l} x_2(0) = 1 \\ x_2(1) = 2 \\ x_2(2) = 1 \\ x_2(3) = 2 \end{array}}$

When $k = 0$; $X_3(0) = X_1(0) \times X_2(0) = 2 \times 6 = 12$

When $k = 1$; $X_3(1) = X_1(1) \times X_2(1) = 0 \times 0 = 0$

When $k = 2$; $X_3(2) = X_1(2) \times X_2(2) = -2 \times -2 = 4$

When $k = 3$; $X_3(3) = X_1(3) \times X_2(3) = 0 \times 0 = 0$

$$\therefore X_3(k) = \{12,\; 0,\; 4,\; 0\}$$

By circular convolution theorem of DFT, we get,

$$\mathcal{DFT}\{x_1(n) \circledast x_2(n)\} = X_1(k)\, X_2(k) \implies x_1(n) \circledast x_2(n) = \mathcal{DFT}^{-1}\{X_1(k)\, X_2(k)\} = \mathcal{DFT}^{-1}\{X_3(k)\}$$

Let $x_3(n)$ be the 4-point sequence obtained by taking inverse DFT of $X_3(k)$.

$$x_3(n) = \mathcal{DFT}^{-1}\{X_3(k)\} = \frac{1}{4}\sum_{k=0}^{4-1} X_3(k)\, e^{\frac{j2\pi kn}{4}} = \frac{1}{4}\sum_{k=0}^{3} X_3(k)\, e^{\frac{j\pi kn}{2}} \;;\; n = 0, 1, 2, 3$$

$$= \frac{1}{4}\left[X_3(0)\, e^0 + X_3(1)\, e^{\frac{j\pi n}{2}} + X_3(2)\, e^{j\pi n} + X_3(3)\, e^{\frac{j3\pi n}{2}}\right]$$

$\boxed{\sin\pi n = 0 \text{ for integer } n}$

$$= \frac{1}{4}\left[12 + 0 + 4e^{j\pi n} + 0\right] = 3 + e^{j\pi n} = 3 + \cos\pi n + j\sin\pi n = 3 + \cos\pi n$$

When $n = 0$; $x_3(0) = 3 + \cos 0 = 3 + 1 = 4$

When $n = 1$; $x_3(1) = 3 + \cos \pi = 3 - 1 = 2$

When $n = 2$; $x_3(2) = 3 + \cos 2\pi = 3 + 1 = 4$

When $n = 3$; $x_3(3) = 3 + \cos 3\pi = 3 - 1 = 2$

$$\therefore\ x_1(n) \circledast x_2(n) = x_3(n) = \{\underset{\uparrow}{4}, 2, 4, 2\}$$

Example 1.10

Compute the linear and circular convolution of the following two sequences using DFT.

$$x(n) = \{\underset{\uparrow}{1}, 2\} \text{ and } h(n) = \{\underset{\uparrow}{2}, 1\}$$

Solution

Linear Convolution by DFT

DFT will not support linear convolution and so linear convolution is performed via circular convolution. The input sequences are converted to size of output sequence of linear convolution by appending zeros. Then the circular convolution of appended sequences will give the result of linear convolution.

Here, x(n) and h(n) are 2 sample sequences. Therefore, the linear convolution of x(n) and h(n) will produce a (2 + 2 – 1 = 3) 3-sample sequence. To avoid time aliasing we convert the 2-sample input sequences into 3-sample sequences by padding with zeros.

$$\boxed{x(0) = 1,\ x(1) = 2,\ x(2) = 0}$$

$$\therefore\ x(n) = \{\underset{\uparrow}{1}, 2, 0\} \text{ and } h(n) = \{\underset{\uparrow}{2}, 1, 0\}$$

By definition of N-point DFT, the 3-point DFT of x(n) is,

$$X(k) = \sum_{n=0}^{3-1} x(n)\, e^{\frac{-j2\pi kn}{3}} = x(0)\,e^0 + x(1)\,e^{\frac{-j2\pi k}{3}} + x(2)\,e^{\frac{-j4\pi k}{3}} = 1 + 2\,e^{\frac{-j2\pi k}{3}}$$

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

When $k = 0$; $X(0) = 1 + 2e^0 = 1 + 2 = 3$

When $k = 1$; $X(1) = 1 + 2e^{\frac{-j2\pi}{3}} = 1 + 2(-0.5 - j0.866) = -j1.732$

When $k = 2$; $X(2) = 1 + 2e^{\frac{-j4\pi}{3}} = 1 + 2(-0.5 + j0.866) = j1.732$

By the definition of N-point DFT, the 3-point DFT of h(n) is,

$$H(k) = \sum_{n=0}^{3-1} h(n)\, e^{\frac{-j2\pi kn}{3}} = h(0)\,e^0 + h(1)\,e^{\frac{-j2\pi k}{3}} + h(2)\,e^{\frac{-j4\pi k}{3}} = 2 + e^{\frac{-j2\pi k}{3}}$$

$$\boxed{h(0) = 2,\ h(1) = 1,\ h(2) = 0}$$

When $k = 0$; $H(0) = 2 + e^0 = 2 + 1 = 3$

When $k = 1$; $H(1) = 2 + e^{\frac{-j2\pi}{3}} = 2 - 0.5 - j0.866 = 1.5 - j0.866$

When $k = 2$; $H(2) = 2 + e^{\frac{-j4\pi}{3}} = 2 - 0.5 + j0.866 = 1.5 + j0.866$

Let $Y(k) = X(k)\, H(k)$; for $k = 0, 1, 2$

When $k = 0$; $Y(0) = X(0)\ H(0) = 3 \times 3 = 9$

When $k = 1$; $Y(1) = X(1)\ H(1) = -j1.732 \times (1.5 - j0.866) = -1.5 - j2.598$

When $k = 2$; $Y(2) = X(2)\ H(2) = j1.732 \times (1.5+j0.866) = -1.5 + j2.598$

$$\therefore Y(k) = \{\underset{\uparrow}{9},\quad -1.5 - j2.598,\quad -1.5 + j2.598\}$$

By circular convolution theorem of DFT, we get,

$$\mathcal{DFT}\{x(n) \circledast h(n)\} = X(k)\,H(k) \implies x(n) \circledast h(n) = \mathcal{DFT}^{-1}\{X(k)\,H(k)\} = \mathcal{DFT}^{-1}\{Y(k)\}$$

Therefore, the sequence y(n) is obtained from inverse DFT of Y(k). By definition of inverse DFT,

$$y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\ e^{\frac{j2\pi kn}{N}} \quad ; \quad \text{for } n = 0, 1, 2, \ldots\ldots.., N-1$$

$$\therefore y(n) = \frac{1}{3}\sum_{k=0}^{2} Y(k)\,e^{\frac{j2\pi kn}{3}} \qquad \boxed{N = 3}$$

$$= \frac{1}{3}\left[Y(0)\,e^{0} + Y(1)\,e^{\frac{j2\pi n}{3}} + Y(2)\,e^{\frac{j4\pi n}{3}}\right] \quad ; \quad \text{for } n = 0, 1, 2$$

$$= \frac{1}{3}\left[9 + (-1.5 - j2.598)\,e^{\frac{j2\pi n}{3}} + (-1.5 + j2.598)\,e^{\frac{j4\pi n}{3}}\right]$$

$$= 3 + (-0.5 - j0.866)\,e^{\frac{j2\pi n}{3}} + (-0.5 + j0.866)\,e^{\frac{j4\pi n}{3}}$$

When $n = 0$; $y(0) = 3 + (-0.5 - j0.866)e^{0} + (-0.5 + j0.866)e^{0}$ $\qquad \boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

$$= 3 - 0.5 - \cancel{j0.866} - 0.5 + \cancel{j0.866} = 2$$

When $n = 1$; $y(1) = 3 + (-0.5 - j0.866)e^{\frac{j2\pi}{3}} + (-0.5 + j0.866)e^{\frac{j4\pi}{3}}$ $\qquad \boxed{(a + b)\,(a - b) = a^2 - b^2}$

$$= 3 + (-0.5 - j0.866)\,(-0.5 + j0.866) + (-0.5 + j0.866)\,(-0.5 - j0.866)$$

$$= 3 + (0.5^2 + 0.866^2) + (0.5^2 + 0.866^2) = 3 + 1 + 1 = 5$$

When $n = 2$; $y(2) = 3 + (-0.5 - j0.866)e^{\frac{j4\pi}{3}} + (-0.5 + j0.866)e^{\frac{j8\pi}{3}}$

$$= 3 + (-0.5 - j0.866)\,(-0.5 - j0.866) + (-0.5 + j0.866)\,(-0.5 + j0.866)$$

$$= 3 + (-0.5 - \cancel{j0.866})^2 + (-0.5 + \cancel{j0.866})^2$$

$$= 3 - 0.5 + j0.866 - 0.5 - j0.866 = 2$$

$$\therefore x(n) * h(n) = y(n) = \{\underset{\uparrow}{2},\ 5,\ 2\}$$

Circular Convolution by DFT

The given sequences are 2-point sequences. Hence, 2-point DFT of the sequences are obtained as follows.

The 2-point DFT of x(n) is given by,

$$X(k) = \sum_{n=0}^{2-1} x(n)\,e^{\frac{-j2\pi kn}{2}} = x(0)\,e^{0} + x(1)\,e^{-j\pi k} = 1 + 2\,e^{-j\pi k} \quad ; \quad \text{for } k = 0, 1 \qquad \boxed{x(0) = 1,\ x(1) = 2}$$

When $k = 0$; $X(0) = 1 + 2\,e^{0} = 1 + 2 = 3$

When $k = 1$; $X(1) = 1 + 2\,e^{-j\pi} = 1 - 2 = -1$

$$\therefore X(k) = \{\underset{\uparrow}{3},\ -1\}$$

The 2-point DFT of h(n) is given by,

$$H(k) = \sum_{n=0}^{2-1} h(n)\, e^{\frac{-j2\pi kn}{2}} = h(0)\, e^{0} + h(1)\, e^{-j\pi k} = 2 + e^{-j\pi k} \quad ; \quad \text{for } k = 0, 1 \qquad \boxed{h(0) = 2,\ h(1) = 1}$$

When k = 0; $H(0) = 2 + e^{0} = 2 + 1 = 3$

When k = 1; $H(1) = 2 + e^{-j\pi} = 2 - 1 = 1$

$$\therefore\ H(k) = \{\underset{\uparrow}{3}, 1\}$$

Let, the product of X(k) and H(k) be equal to Y(k).

$$\therefore\ Y(k) = X(k)\, H(k)\ ; \quad \text{for } k = 0, 1$$

When $k = 0$; $Y(0) = X(0)\ H(0) = 3 \times 3 = 9$

When $k = 1$; $Y(1) = X(1)\ H(1) = -1 \times 1 = -1$

$$\therefore\ Y(k) = \{\underset{\uparrow}{9}, -1\}$$

By circular convolution theorem of DFT we get,

$$\mathcal{DFT}\{x(n) \circledast h(n)\} = X(k)\, H(k) \ \Rightarrow\ x(n) \circledast h(n) = \mathcal{DFT}^{-1}\{X(k)\, H(k)\} = \mathcal{DFT}^{-1}\{Y(k)\}$$

Therefore, the sequence y(n) is obtained from inverse DFT of Y(k). By the definition of inverse DFT,

$$y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \frac{1}{N} \sum_{k=0}^{N-1} Y(k)\, e^{\frac{j2\pi kn}{N}} \quad ; \quad \text{for } n = 0, 1, 2, \ldots\ldots, N-1 \qquad \boxed{N = 2}$$

$$\therefore\ y(n) = \frac{1}{2} \sum_{k=0}^{1} Y(k)\, e^{\frac{j2\pi kn}{2}} = \frac{1}{2}\left[Y(0) + Y(1)\, e^{j\pi n}\right] = \frac{1}{2}\left[9 - e^{j\pi n}\right] = 4.5 - 0.5 e^{j\pi n}$$

$$\boxed{e^{j\pi} = -1}$$

When $n = 0$; $y(0) = 4.5 - 0.5 e^{0} = 4.5 - 0.5 = 4$

When $n = 1$; $y(1) = 4.5 - 0.5 e^{j\pi} = 4.5 + 0.5 = 5$

$$\therefore\ x(n) \circledast y(n) = y(n) = \{\underset{\uparrow}{4}, 5\}$$

Example 1.11

Using DFT determine the response of FIR filter with impulse response, h(n) = {1, 2} for the input x(n) = {2, –1, 2}.

Solution

The response of FIR filter is given by convolution of input and impulse response.

Let, $y(n) = x(n) * h(n)$ = Response of FIR filter.

Here, x(n) is 3-point sequence and h(n) is 2-point sequence.

Let, M = Length of y(n)

Now, M = 3 + 2 – 1 = 4

Since convolution is performed via DFT let us convert x(n) and h(n) to M-point sequences by appending zeros.

$\therefore$ x(n) = {2, –1, 2, 0}

h(n) = {1, 2, 0, 0}

M-point DFT of x(n)

By the definition of DFT

$$X(k) = \mathcal{DFT}\{x(n)\} = \sum_{n=0}^{M-1} x(n)\, e^{-\frac{j2\pi nk}{N}} \quad ; \quad \text{for } k = 0, 1, 2, 3$$

$$= \sum_{n=0}^{3} x(n)\, e^{-\frac{j2\pi nk}{4}} = \sum_{n=0}^{3} x(n)\, e^{-\frac{j\pi nk}{2}}$$

$$= 2 \times e^{0} - 1 \times e^{-\frac{j\pi k}{2}} + 2 \times e^{-j\pi k} + 0$$

$$= 2 - e^{-\frac{j\pi k}{2}} + 2e^{-j\pi k}$$

$x(0) = 2$, $x(1) = -1$
$x(2) = 2$, $x(3) = 0$

$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

$e^{\pm\frac{j\pi}{2}} = \pm j$
$e^{\pm j\pi} = e^{\pm j3\pi} = -1$
$e^{\pm j2\pi} = 1$
$e^{\pm\frac{j3\pi}{2}} = \mp j$

When $k = 0$, $X(0) = 2 - e^{0} + 2e^{0} = 2 - 1 + 2 = 3$

When $k = 1$, $X(1) = 2 - e^{-\frac{j\pi}{2}} + 2e^{-j\pi} = 2 + j - 2 = j$

When $k = 2$, $X(2) = 2 - e^{-j\pi} + 2e^{-j2\pi} = 2 + 1 + 2 = 5$

When $k = 3$, $X(3) = 2 - e^{-\frac{j3\pi}{2}} + 2e^{-j3\pi} = 2 - j - 2 = -j$

$\therefore\ X(k) = \{3,\ j,\ 5,\ -j\}$

M-point DFT of h(n)

By the definition of DFT

$$H(k) = \mathcal{DFT}\{h(n)\} = \sum_{n=0}^{M-1} h(n)\, e^{-\frac{j2\pi nk}{M}} \quad ; \quad \text{for } k = 0, 1, 2, 3$$

$$= \sum_{n=0}^{3} h(n)\, e^{-\frac{j2\pi nk}{4}} = \sum_{n=0}^{3} h(n)\, e^{-\frac{j\pi nk}{2}}$$

$$= 2 \times e^{0} + 2 \times e^{-\frac{j\pi k}{2}} + 0 + 0$$

$$= 1 + 2e^{-\frac{j\pi k}{2}}$$

$h(0) = 1$, $h(1) = 2$
$h(2) = h(3) = 0$

When $k = 0$, $H(0) = 1 + 2e^{0} = 1 + 2 = 3$

When $k = 1$, $H(1) = 1 + 2e^{-\frac{j\pi}{2}} = 1 - 2j$

When $k = 2$, $H(2) = 1 + 2e^{-j\pi} = 1 - 2 = -1$

When $k = 3$, $H(3) = 1 + 2e^{-\frac{j3\pi}{2}} = 1 + 2j$

$\therefore\ H(k) = \{3,\ 1 - 2j,\ -1,\ 1 + 2j\}$

Product of X(k) and H(k)

Let, $Y(k) = X(k)\, H(k)$

When $k = 0$, $Y(0) = X(0) \times H(0) = 3 \times 3 = 9$

When $k = 1$, $Y(1) = X(1) \times H(1) = j \times (1 - 2j) = 2 + j$

When $k = 2$, $Y(2) = X(2) \times H(2) = 5 \times (-1) = -5$

When $k = 3$, $Y(3) = X(3) \times H(3) = -j \times (1 + 2j) = 2 - j$

$\therefore\ Y(k) = \{9,\ 2 + j,\ -5,\ 2 - j\}$

Response of the FIR filter

The response y(n) of FIR filter is given by inverse DFT of Y(k).

By definition of inverse DFT,

$$y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \frac{1}{M}\sum_{k=0}^{M-1} Y(k)\, e^{\frac{j2\pi nk}{M}} \quad ; \quad \text{for } n = 0, 1, 2, 3$$

$$= \frac{1}{4}\sum_{k=0}^{3} Y(k)\, e^{\frac{j2\pi nk}{4}} = \frac{1}{4}\sum_{k=0}^{3} Y(k)\, e^{\frac{j\pi nk}{2}}$$

$$= \frac{1}{4}\left[9 \times e^{0} + (2+j)\, e^{\frac{j\pi n}{2}} - 5e^{j\pi n} + (2-j)\, e^{\frac{j3\pi n}{2}}\right]$$

$$= \frac{1}{4}\left[9 + (2+j)\, e^{\frac{j\pi n}{2}} - 5e^{j\pi n} + (2-j)\, e^{\frac{j3\pi n}{2}}\right]$$

When $n = 0$, $y(0) = \frac{1}{4}\left[9 + (2+j)\, e^{0} - 5e^{0} + (2-j)\, e^{0}\right] = \frac{1}{4}[9 + 2 + j - 5 + 2 - j] = \frac{8}{4} = 2$

When $n = 1$, $y(1) = \frac{1}{4}\left[9 + (2+j)\, e^{\frac{j\pi}{2}} - 5e^{j\pi} + (2-j)\, e^{\frac{j3\pi}{2}}\right]$

$$= \frac{1}{4}[9 + (2+j) \times j - 5 \times (-1) + (2-j) \times (-j)] = \frac{1}{4}[9 + 2j - 1 + 5 - 2j - 1] = \frac{12}{4} = 3$$

When $n = 2$, $y(2) = \frac{1}{4}\left[9 + (2+j)\, e^{j\pi} - 5e^{j2\pi} + (2-j)\, e^{j3\pi}\right]$

$$= \frac{1}{4}[9 + (2+j) \times (-1) - 5 \times 1 + (2-j) \times (-1)] = \frac{1}{4}[9 - 2 - j - 5 - 2 - j)] = \frac{0}{4} = 0$$

When $n = 3$, $y(3) = \frac{1}{4}\left[9 + (2+j)\, e^{\frac{j3\pi}{2}} - 5e^{j3\pi} + (2-j)\, e^{\frac{j9\pi}{2}}\right]$ $\boxed{e^{\frac{j9\pi}{2}} = j}$

$$= \frac{1}{4}[9 + (2+j)(-j) - 5 \times (-1) + (2-j)(j)] = \frac{1}{4}[9 - 2j + 1 + 5 + 2j + 1] = \frac{16}{4} = 4$$

$\therefore$ Response, $y(n) = \{2,\ 3,\ 0,\ 4\}$

1.8 Fast Computation of DFT (or Fast Fourier Transform (FFT))

Fourier Transform (FFT): *It is a method (or algorithm) for computing the Discrete Fourier Transform (DFT) with reduced number of calculations.*

The computational efficiency is achieved if we adopt a divide and conquer approach. This approach is based on the decomposition of an N-point DFT into successively smaller DFTs. This basic approach leads to a family of an efficient computational algorithms collectively known as FFT algorithms.

Radix-r FFT

In an N-point sequence, if N can be expressed as $N = r^{m}$, then the sequence can be decimated into r-point sequences. For each r-point sequence, r-point DFT can be computed. From the results of r-point DFT, the r^{2}-point DFTs are computed.

From the results of r^{2}-point DFTs, the r^{3}-point DFTs are computed and so on, until we get r^{m} point DFT. This FFT algorithm is called radix-**r** FFT. In computing N-point DFT by this method the number of stages of computation will be **m** times.

Radix-2 FFT

For radix-2 FFT, the value of N should be such that $N = 2^m$, so that the N-point sequence is decimated into 2-point sequences and the 2-point DFT for each decimated sequence is computed. From the results of 2-point DFTs, the 4-point DFTs can be computed. From the results of 4-point DFTs, the 8-point DFTs can be computed and so on, until we get N-point DFT.

Number of Calculations in N-point DFT

Let X(k) be N-point DFT of an L-point discrete time sequence x(n), where $N \geq L$. Now, the N-point DFT is a sequence consisting of N-complex numbers. Each complex number of the sequence is calculated using the following equation (equation 1.39).

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad \text{for } k = 0, 1, 2, \text{.......}, N-1$$

$$= x(0)\, e^0 + x(1)\, e^{\frac{-j2\pi k}{N}} + x(2)\, e^{\frac{-j4\pi k}{N}} + x(3)\, e^{\frac{-j6\pi k}{N}} + \text{..............} + x(N-1)\, e^{\frac{-j2(N-1)\pi k}{N}}$$

$$\therefore X(k) = \underbrace{\underbrace{x(0)e^0}_{\substack{\text{Complex}\\ \text{multiplication}}} + \underbrace{x(1)e^{\frac{-j2\pi k}{N}}}_{\substack{\text{Complex}\\ \text{multiplication}}} + \underbrace{x(2)e^{\frac{-j4\pi k}{N}}}_{\substack{\text{Complex}\\ \text{multiplication}}} + \underbrace{x(3)e^{\frac{-j6\pi k}{N}}}_{\substack{\text{Complex}\\ \text{multiplication}}} + \text{..........} + \underbrace{x(N-1)e^{\frac{-j2(N-1)\pi k}{N}}}_{\substack{\text{Complex}\\ \text{multiplication}}}}_{N-1\ \text{Complex additions}}$$

From the above equation we can say that,

The number of calculations to calculate X(k) for one value of k are,

N number of complex multiplications and

N – 1 number of complex additions.

The X(k) is a sequence consisting of N complex numbers.

Therefore, the number of calculations to calculate all the N complex numbers of the X(k) are,

$N \times N = N^2$ number of complex multiplications and

$N \times (N-1) = N(N-1)$ number of complex additions.

Hence, in direct computation of N-point DFT, the total number of complex additions are $N(N-1)$ and total number of complex multiplications are N^2.

Number of Calculations in Radix-2 FFT

In radix-2 FFT, $N = 2^m$, and so there will be m stages of computations, where $m = \log_2 N$, with each stage having N/2 butterflies. (Refer section 1.9.2 and 1.10.2).

The number of calculations in one butterfly are,

1 number of Complex multiplication and

2 number of Complex additions.

There are $\frac{N}{2}$ butterflies in each stage.

Therefore, number of calculations in one stage are,

$$\frac{N}{2} \times 1 = \frac{N}{2} \text{ Complex multiplications and}$$

$$\frac{N}{2} \times 2 = N \text{ Complex additions}$$

The N-point DFT involves m stages of computations. Therefore, the number of calculations for m stages are,

$$m \times \frac{N}{2} = \log_2 N \times \frac{N}{2} = \frac{N}{2} \log_2 N \text{ Complex multiplications and}$$

$$m \times N = \log_2 N \times N = N\log_2 N \text{ Complex additions.}$$

$\boxed{N = 2^m}$

$\boxed{\log_2 N = \log_2 2^m = m}$

Hence, in radix-2 FFT, the total number of complex additions are reduced to $N\log_2 N$ and total number of complex multiplications are reduced to $(N/2) \log_2 N$.

Table 1.5 presents a comparison of the number of complex multiplications and additions in radix-2 FFT and in direct computation of DFT. From the table it can be observed that for large values of N, the percentage reduction in calculations is also very large.

$\boxed{\log_y x = \frac{\log_{10} x}{\log_{10} y}}$

$\boxed{\log_2 2^m = m}$

Table 1.5: Comparison of Number of Computation in Direct DFT and FFT

	Direct Computation		**Radix-2 FFT**	
Number of Points N	**Complex additions $N(N-1)$**	**Complex Multiplications N^2**	**Complex additions $N\log_2 N$**	**Complex Multiplication $(N/2) \log_2 N$**
$4 (= 2^2)$	12	16	$4 \times \log_2 2^2 = 4 \times 2 = 8$	$\frac{4}{2} \times \log_2 2^2 = \frac{4}{2} \times 2 = 4$
$8 (= 2^3)$	56	64	$8 \times \log_2 2^3 = 8 \times 3 = 24$	$\frac{8}{2} \times \log_2 2^3 = \frac{8}{2} \times 3 = 12$
$16 (= 2^4)$	240	256	$16 \times \log_2 2^4 = 16 \times 4 = 64$	$\frac{16}{2} \times \log_2 2^4 = \frac{16}{2} \times 4 = 32$
$32 (= 2^5)$	992	1,024	$32 \times \log_2 2^5 = 32 \times 5 = 160$	$\frac{32}{2} \times \log_2 2^5 = \frac{32}{2} \times 5 = 80$
$64 (= 2^6)$	4,032	4,096	$64 \times \log_2 2^6 = 64 \times 6 = 384$	$\frac{64}{2} \times \log_2 2^6 = \frac{64}{2} \times 6 = 192$
$128 (= 2^7)$	16,256	16,384	$128 \times \log_2 2^7 = 128 \times 7 = 896$	$\frac{128}{2} \times \log_2 2^7 = \frac{128}{2} \times 7 = 448$

Phase or Twiddle Factor

By the definition of DFT, the N-point DFT is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad \text{for } k = 0, 1, 2, ..., N-1 \qquad .....(1.68)$$

To simplify the notation it is desirable to define the complex valued phase factor W_N (also called as twiddle factor) which is an N^{th} root of unity as,

$$W_N = e^{\frac{-j2\pi}{N}}$$

Here, W represents a complex number $1\angle -2\pi$. Hence, the phase or argument of W is -2π. Therefore, when a number is multiplied by W, only its phase changes by -2π but magnitude remains the same.

$$\therefore\ W = e^{-j2\pi}$$

The phase value -2π of W can be multiplied by any integer and it is represented as prefix in W. For example, multiplying -2π by k can be represented as W^k.

$$\therefore\ e^{-j2\pi\times k} \Rightarrow W^k$$

The phase value -2π of W can be divided by any integer and it is represented as suffix in W. For example, dividing -2π by N can be represented as W_N.

$$\therefore\ e^{-j2\pi\div N} = e^{-j2\pi\times\frac{1}{N}} \Rightarrow W_N$$

$$\therefore\ e^{\frac{-j2\pi nk}{N}} = e^{-j2\pi\times\frac{nk}{N}} = W_N^{nk} \qquad .....(1.69)$$

Using equation (1.69), the equation (1.68) can be written as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \ ; \quad \text{for } k = 0, 1, 2, \ ...., \ N-1 \qquad .....(1.70)$$

Equation (1.70) is the definition of N-point DFT using phase factor and this equation is popularly used in FFT.

1.9 Radix-2 Decimation in Time (DIT) Fast Fourier Transform (FFT)

The N-point DFT of a sequence x(n) converts the time domain N-point sequence x(n) into a frequency domain N-point sequence X(k). In **Decimation In Time (DIT)** algorithm, the time domain sequence x(n) is decimated and smaller point DFTs are performed. The results of smaller point DFTs are combined to get the result of N-point DFT.

In DIT radix-2 FFT, the time domain sequence is decimated into 2-point sequences. For each two point sequence, a 2-point DFT is computed. The results of 2-point DFTs are used to compute 4-point DFTs. A pair of 2-point DFT results are used to compute one 4-point DFT. The results of 4-point DFTs are used to compute 8-point DFTs. This process is continued until we get N-point DFT.

In general, we can say that in decimation in time algorithm, the N-point DFT can be realised from two numbers of N/2 point DFTs, the N/2 point DFT can be realised from two numbers of N/4 point DFTs and so on.

Let, x(n) be N-sample sequence. We can decimate x(n) into two sequences of N/2 samples. Let the two sequences be $f_1(n)$ and $f_2(n)$. Let $f_1(n)$ consists of even numbered samples of x(n) and $f_2(n)$ consists of odd numbered samples of x(n).

$$\therefore f_1(n) = x(2n) \quad ; \text{ for } n = 0, 1, 2, 3 \,........, \frac{N}{2} - 1$$

$$\therefore f_2(n) = x(2n+1)\,; \text{ for } n = 0, 1, 2, 3 \,........, \frac{N}{2} - 1$$

Let, $X(k)$ = N-point DFT of $x(n)$

$F_1(k)$ = N/2 point DFT of $f_1(n)$

$F_2(k)$ = N/2 point DFT of $f_2(n)$

By definition of DFT, the N/2 point DFT of $f_1(n)$ and $f_2(n)$ are given by,

$$F_1(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n)\, W_{\frac{N}{2}}^{kn} \quad ; \quad F_2(k) = \sum_{n=0}^{\frac{N}{2}-1} f_2(n)\, W_{\frac{N}{2}}^{kn}$$

Now, N-point DFT $X(k)$, in terms of N/2 point DFTs $F_1(k)$ and $F_2(k)$ is given by,

$$X(k) = F_1(k) + W_N^k\, F_2(k), \quad \text{where, } k = 0, 1, 2, 3, ..., N-1 \qquad .....(1.71)$$

The proof of equation (1.71) is given below.

Proof:

When n is replaced by 2n,even numbered sample of x(n) are selected.
When n is replaced by 2n + 1, odd numbered sample of x(n) are selected.

By definition of DFT, the N-point DFT of x(n) is,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=\text{even}} x(n)\, W_N^{kn} + \sum_{n=\text{odd}} x(n)\, W_N^{kn} \quad ; \; k = 0, 1, 2, ..., N-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_N^{k(2n)} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_N^{k(2n+1)} \qquad .....(1.72)$$

The phase factors in equation (1.72) can be modified as shown below.

$$W_N^{k(2n)} = (e^{-j2\pi})^{\frac{k(2n)}{N}} = (e^{-j2\pi})^{\frac{kn}{N/2}} = W_{N/2}^{kn} \qquad .....(1.73)$$

$$W_N^{k(2n+1)} = (e^{-j2\pi})^{\frac{k(2n+1)}{N}} = (e^{-j2\pi})^{\frac{k2n}{N}}(e^{-j2\pi})^{\frac{k}{N}} = (e^{-j2\pi})^{\frac{kn}{N/2}}(e^{-j2\pi})^{\frac{k}{N}} = W_{N/2}^{kn} W_N^{k} \qquad .....(1.74)$$

Using equations (1.73) and (1.74), equation (1.72) can be written as,

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_{N/2}^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_{N/2}^{kn} W_N^{k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f_1(n)\, W_{N/2}^{kn} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} f_2(n)\, W_{N/2}^{kn} \qquad .....(1.75)$$

$x(2n) = f_1(n)$ and $x(2n+1) = f_2(n)$

By definition of DFT the N/2 point DFT of $f_1(n)$ and $f_2(n)$ are given by,

$$F_1(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n)\, W_{N/2}^{kn} \quad \text{and} \quad F_2(k) = \sum_{n=0}^{\frac{N}{2}-1} f_2(n) W_{N/2}^{kn} \qquad .....(1.76)$$

Using equation (1.76) in equation (1.75) we get,

$$X(k) = F_1(k) + W_N^k\, F_2(k), \quad \text{where, } k = 0, 1, 2, 3, ..., N-1$$

Having performed the decimation in time once, we can repeat the process for each of the sequences $f_1(n)$ and $f_2(n)$. Thus, $f_1(n)$ would result in two N/4 point sequences and $f_2(n)$ would result in another two N/4 point sequences.

Let the decimated N/4 point sequences of $f_1(n)$ be $v_{11}(n)$ and $v_{12}(n)$.

$$\therefore v_{11}(n) = f_1(2n) \quad ; \quad \text{for } n = 0, 1, 2, ..., \frac{N}{4} - 1$$

$$v_{12}(n) = f_1(2n+1) \quad ; \quad \text{for } n = 0, 1, 2, ..., \frac{N}{4} - 1$$

Let the decimated N/4 point sequences of $f_2(n)$ be $v_{21}(n)$ and $v_{22}(n)$.

$$\therefore v_{21}(n) = f_2(2n) \quad ; \quad \text{for } n = 0, 1, 2, ..., \frac{N}{4} - 1$$

$$v_{22}(n) = f_2(2n+1) \quad ; \quad \text{for } n = 0, 1, 2, ..., \frac{N}{4} - 1$$

Let, $V_{11}(k)$ = N/4 point DFT of $v_{11}(n)$; $V_{21}(k)$ = N/4 point DFT of $v_{21}(n)$

$V_{12}(k)$ = N/4 point DFT of $v_{12}(n)$; $V_{22}(k)$ = N/4 point DFT of $v_{22}(n)$

Then like the earlier analysis we can show that,

$$F_1(k) = V_{11}(k) + W_{N/2}^{k} V_{12}(k), \quad \text{for } k = 0, 1, 2, 3, ..., \frac{N}{2} - 1 \qquad .....(1.77)$$

$$F_2(k) = V_{21}(k) + W_{N/2}^{k} V_{22}(k), \quad \text{for } k = 0, 1, 2, 3, ..., \frac{N}{2} - 1 \qquad .....(1.78)$$

Hence, the N/2 point DFTs are obtained from the results of N/4 point DFTs.

The decimation of the data sequence can be repeated until the resulting sequences are reduced to 2-point sequences.

1.9.1 8-Point DFT using Radix-2 DIT FFT

The input sequence is 8-point sequence. Therefore, $N = 8 = 2^3 = r^m$. Here, r = 2 and m = 3.

Therefore, the computation of 8-point DFT using radix-2 FFT involves three stages of computation. The given 8-point sequence is decimated to 2-point sequences.

For each 2-point sequence, the 2-point DFT is computed. From the results of 2-point DFT, the 4-point DFT is computed. From the results of 4-point DFT, the 8-point DFT is computed.

Let the given sequence be x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7), which consists of 8 samples. The 8-samples should be decimated into sequences of 2-samples. Before decimation they are arranged in bit reversed order, as shown in Table 1.6.

The x(n) in bit reversed order is decimated into 4 numbers of 2-point sequences as shown ahead.

Table 1.6

Normal order		**Bit reversed order**	
x(0)	x(000)	x(0)	x(000)
x(1)	x(001)	x(4)	x(100)
x(2)	x(010)	x(2)	x(010)
x(3)	x(011)	x(6)	x(110)
x(4)	x(100)	x(1)	x(001)
x(5)	x(101)	x(5)	x(101)
x(6)	x(110)	x(3)	x(011)
x(7)	x(111)	x(7)	x(111)

***Note:** In a discrete sequence x(n), if the samples of the sequence are arranged such that the binary representation of n is mirror image of original binary representation then the sequence is set to be in bit reversed order.*

Sequence-1: {x(0), x(4)}

Sequence-2: {x(2), x(6)}

Sequence-3: {x(1), x(5)}

Sequence-4: {x(3), x(7)}

Using the decimated sequences as input, the 8-point DFT is computed. Fig 1.37 shows the three stages of computation of an 8-point DFT.

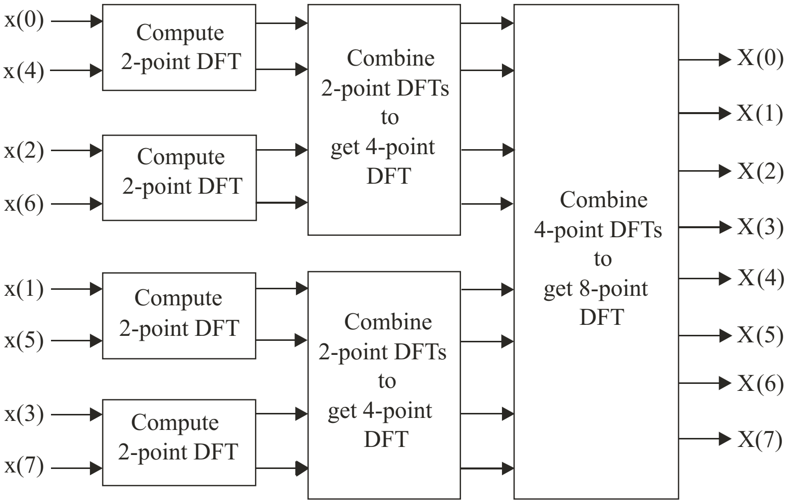


***Fig 1.37:** Three stages of computations of an 8-point DFT*

Let us examine the 8-point DFT of an 8-point sequence in detail. The 8-point sequence is decimated into 4-point sequences and 2-point sequences as shown below:

Let, x(n) = 8-point sequence

$f_1(n), f_2(n)$ = 4-point sequences obtained from x(n)

$v_{11}(n), v_{12}(n)$ = 2-point sequences obtained from $f_1(n)$

$v_{21}(n), v_{22}(n)$ = 2-point sequences obtained from $f_2(n)$.

The relations between the samples of various sequences are given below.

$v_{11}(0) = f_1(0) = x(0)$

$v_{11}(1) = f_1(2) = x(4)$

$v_{12}(0) = f_1(1) = x(2)$

$v_{12}(1) = f_1(3) = x(6)$

$v_{21}(0) = f_2(0) = x(1)$

$v_{21}(1) = f_2(2) = x(5)$

$v_{22}(0) = f_2(1) = x(3)$

$v_{22}(1) = f_2(3) = x(7)$

First Stage Computation

In the first stage of computation, the two point DFTs of the 2-point sequences are computed.

Let, $V_{11}(k) = \mathcal{DFT}\{v_{11}(n)\}$.

Using equation (1.70), the 2-point DFT of $v_{11}(n)$ is given by,

$$V_{11}(k) = \sum_{n=0,1} v_{11}(n)\,W_2^{nk} = v_{11}(0)\,W_2^0 + v_{11}(1)\,W_2^k \;;\; \text{for } k = 0, 1$$

When k = 0 ; $V_{11}(k) = V_{11}(0) = v_{11}(0)W_2^0 + v_{11}(1)W_2^0 = v_{11}(0) + v_{11}(1) = x(0) + x(4)$

When k = 1 ; $V_{11}(k) = V_{11}(1) = v_{11}(0)W_2^0 + v_{11}(1)W_2^1 = v_{11}(0) - W_2^0 v_{11}(1) = x(0) - W_2^0 x(4)$

$$\boxed{W_2^0 = e^{j2\pi\times\frac{0}{2}} = e^0 = 1}\;\boxed{W_2^1 = e^{-j2\pi\times\frac{1}{2}} = e^{-j\pi} = (\cos\pi - j\sin\pi) = -1 = -1\times W_2^0 = -W_2^0}$$

Let, $V_{12}(k) = \mathcal{DFT}\{v_{12}(n)\}$.

Using equation (1.70), the 2-point DFT of $v_{12}(n)$ is given by,

$$V_{12}(k) = \sum_{n=0,1} v_{12}(n)\,W_2^{nk} = v_{12}(0)\,W_2^0 + v_{12}(1)\,W_2^k \;;\; \text{for } k = 0, 1$$

When k = 0 ; $V_{12}(k) = V_{12}(0) = v_{12}(0)W_2^0 + v_{12}(1)W_2^0 = v_{12}(0) + v_{12}(1) = x(2) + x(6)$

When k = 1 ; $V_{12}(k) = V_{12}(1) = v_{12}(0)W_2^0 + v_{12}(1)W_2^1 = v_{12}(0) - W_2^0 v_{12}(1) = x(2) - W_2^0 x(6)$

Let, $V_{21}(k) = \mathcal{DFT}\{v_{21}(n)\}$.

Using equation (1.70), the 2-point DFT of $v_{21}(n)$ is given by,

$$V_{21}(k) = \sum_{n=0,1} v_{21}(n)\,W_2^{nk} = v_{21}(0)\,W_2^0 + v_{21}(1)\,W_2^k \;;\; \text{for } k = 0, 1$$

When k = 0 ; $V_{21}(k) = V_{21}(0) = v_{21}(0)W_2^0 + v_{21}(1)W_2^0 = v_{21}(0) + v_{21}(1) = x(1) + x(5)$

When k = 1 ; $V_{21}(k) = V_{21}(1) = v_{21}(0)W_2^0 + v_{21}(1)W_2^1 = v_{21}(0) - W_2^0 v_{21}(1) = x(1) - W_2^0 x(5)$

Let, $V_{22}(k) = \mathcal{DFT}\{v_{22}(n)\}$.

Using equation (1.70), the 2-point DFT of $v_{22}(n)$ is given by,

$$V_{22}(k) = \sum_{n=0,1} v_{22}(n)\,W_2^{nk} = v_{22}(0)\,W_2^0 + v_{22}(1)\,W_2^k \;;\; \text{for } k = 0, 1$$

When k = 0 ; $V_{22}(k) = V_{22}(0) = v_{22}(0)W_2^0 + v_{22}(1)W_2^0 = v_{22}(0) + v_{22}(1) = x(3) + x(7)$

When k = 1 ; $V_{22}(k) = V_{22}(1) = v_{22}(0)W_2^0 + v_{22}(1)W_2^1 = v_{22}(0) - W_2^0 v_{22}(1) = x(3) - W_2^0 x(7)$

Second Stage Computation

In the second stage of computation, the 4-point DFTs are computed using the results of the first stage as input. Let, $F_1(k) = \mathcal{DFT}\{f_1(n)\}$. The 4-point DFT of $f_1(n)$ can be computed using equation (1.77).

$$\therefore F_1(k) = V_{11}(k) + W_4^k V_{12}(k) \ ; \text{ for } k = 0, 1, 2, 3.$$

$V_{11}(k)$ and $V_{12}(k)$ are periodic with periodicity of 2 samples.
$\therefore V_{11}(k+2) = V_{11}(k)$
$V_{12}(k+2) = V_{12}(k)$

When k = 0 ; $F_1(k) = F_1(0) = V_{11}(0) + W_4^0 V_{12}(0)$

When k = 1 ; $F_1(k) = F_1(1) = V_{11}(1) + W_4^1 V_{12}(1)$

When k = 2 ; $F_1(k) = F_1(2) = V_{11}(2) + W_4^2 V_{12}(2) = V_{11}(0) - W_4^0 V_{12}(0)$

When k = 3 ; $F_1(k) = F_1(3) = V_{11}(3) + W_4^3 V_{12}(3) = V_{11}(1) - W_4^1 V_{12}(1)$

$$\boxed{W_4^2 = e^{-j2\pi\times\frac{2}{4}} = e^{-j\pi} = (\cos\pi - j\sin\pi) = -1 = -1\times W_4^0 = -W_4^0}$$

$$\boxed{W_4^3 = e^{-j2\pi\times\frac{3}{4}} = e^{-j2\pi\times\frac{2}{4}}e^{-j2\pi\times\frac{1}{4}} = e^{-j\pi}e^{-j2\pi\times\frac{1}{4}} = (\cos\pi - j\sin\pi)W_4^1 = -1\times W_4^1 = -W_4^1}$$

Let, $F_2(k) = \mathcal{DFT}\{f_2(n)\}$. The 4-point DFT of $f_2(n)$ can be computed using equation (1.78).

$$\therefore F_2(k) = V_{21}(k) + W_4^k V_{22}(k) \ ; \text{ for } k = 0, 1, 2, 3.$$

$V_{21}(k)$ and $V_{22}(k)$ are periodic with periodicity of 2 samples.
$\therefore V_{21}(k+2) = V_{21}(k)$
$V_{22}(k+2) = V_{22}(k)$

When k = 0 ; $F_2(k) = F_2(0) = V_{21}(0) + W_4^0 V_{22}(0)$

When k = 1 ; $F_2(k) = F_2(1) = V_{21}(1) + W_4^1 V_{22}(1)$

When k = 2 ; $F_2(k) = F_2(2) = V_{21}(2) + W_4^2 V_{22}(2) = V_{21}(0) - W_4^0 V_{22}(0)$

When k = 3 ; $F_2(k) = F_2(3) = V_{21}(3) + W_4^3 V_{22}(3) = V_{21}(1) - W_4^1 V_{22}(1)$

Third Stage Computation

In the third stage of computation, the 8-point DFTs are computed using the results of the second stage as inputs.

Let, $X(k) = \mathcal{DFT}\{X(n)\}$. The 8-point DFT of x(n) can be computed using equation (1.71).

$$\therefore X(k) = F_1(k) + W_8^k F_2(k) \ ; \text{ for } k = 0, 1, 2, 3, 4, 5, 6, 7$$

$F_1(k)$ and $F_2(k)$ are periodic with periodicity of 4 samples.
$\therefore F_1(k+4) = F_1(k)$
$F_2(k+4) = F_2(k)$

When k = 0 ; $X(k) = X(0) = F_1(0) + W_8^0 F_2(0)$

When k = 1 ; $X(k) = X(1) = F_1(1) + W_8^1 F_2(1)$

When k = 2 ; $X(k) = X(2) = F_1(2) + W_8^2 F_2(2)$

When k = 3 ; $X(k) = X(3) = F_1(3) + W_8^3 F_2(3)$

When k = 4 ; $X(k) = X(4) = F_1(4) + W_8^4 F_2(4) = F_1(0) - W_8^0 F_2(0)$

When k = 5 ; $X(k) = X(5) = F_1(5) + W_8^5 F_2(5) = F_1(1) - W_8^1 F_2(1)$

When k = 6 ; $X(k) = X(6) = F_1(6) + W_8^6 F_2(6) = F_1(2) - W_8^2 F_2(2)$

When k = 7 ; $X(k) = X(7) = F_1(7) + W_8^7 F_2(7) = F_1(3) - W_8^3 F_2(3)$

$$\boxed{\begin{aligned} W_8^4 &= e^{-j2\pi\times\frac{4}{8}} = e^{-j\pi} \\ &= (\cos\pi - j\sin\pi) \\ &= -1 \end{aligned}}$$

$$\boxed{W_8^4 = W_8^4\times W_8^0 = -W_8^0} \quad \boxed{W_8^5 = W_8^4\times W_8^1 = -W_8^1} \quad \boxed{W_8^6 = W_8^4\times W_8^2 = -W_8^2} \quad \boxed{W_8^7 = W_8^4\times W_8^3 = -W_8^3}$$

1.9.2 Flow Graph for 8-Point DFT using Radix-2 DIT FFT

If we observe the basic computation performed at every stage of radix-2 DIT FFT in the previous section, we can arrive at the following conclusion.

1. In each computation two complex numbers "a" and "b" are considered.
2. The complex number "b" is multiplied by a phase factor "W_N^k".
3. The product "bW_N^k" is added to the complex number "a" to form a new complex number "A".
4. The product "bW_N^k" is subtracted from the complex number "a" to form a new complex number "B".

The above basic computation can be expressed by a signal flow graph shown in Fig 1.38. (For detailed discussion on signal flow graph, refer Section 1.1.12).

The signal flow graph is also called a ***butterfly diagram*** since it resembles a butterfly. In radix-2 FFT, N/2 butterflies per stage are required to represent the computational process. The butterfly diagram used to compute the 8-point DFT via radix-2 DIT FFT can be arrived as shown below, using the computations shown in the previous section.

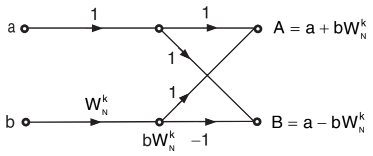


Fig 1.38: *Basic butterfly or flow graph of DIT radix-2 FFT.*

The sequence x(n) is arranged in bit reversed order and then decimated into two sample sequences as shown below.

x(0)	x(2)	x(1)	x(3)
x(4)	x(6)	x(5)	x(7)

Flow Graph or (Butterfly Diagram) for First Stage of Computation

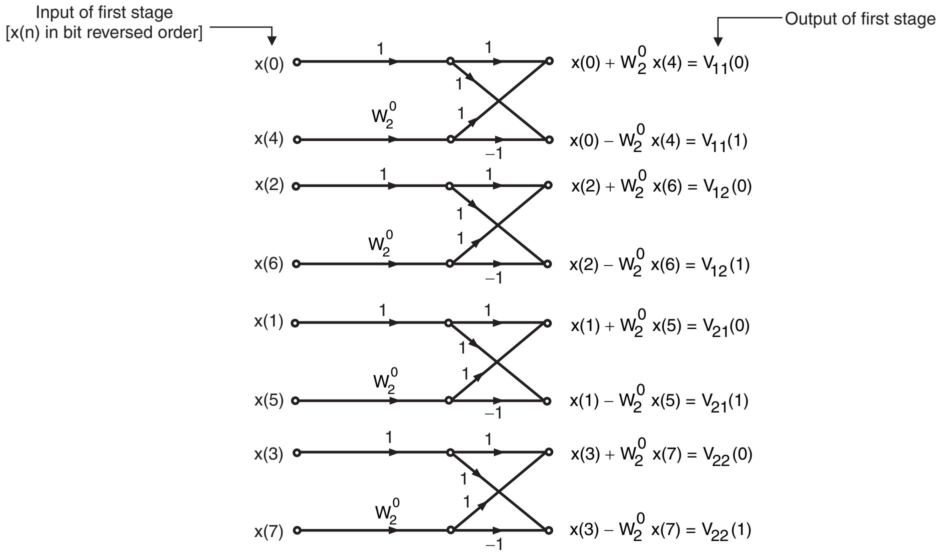


Fig 1.39: *First stage of flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIT FFT.*

Flow Graph (or Butterfly Diagram) for Second Stage of Computation

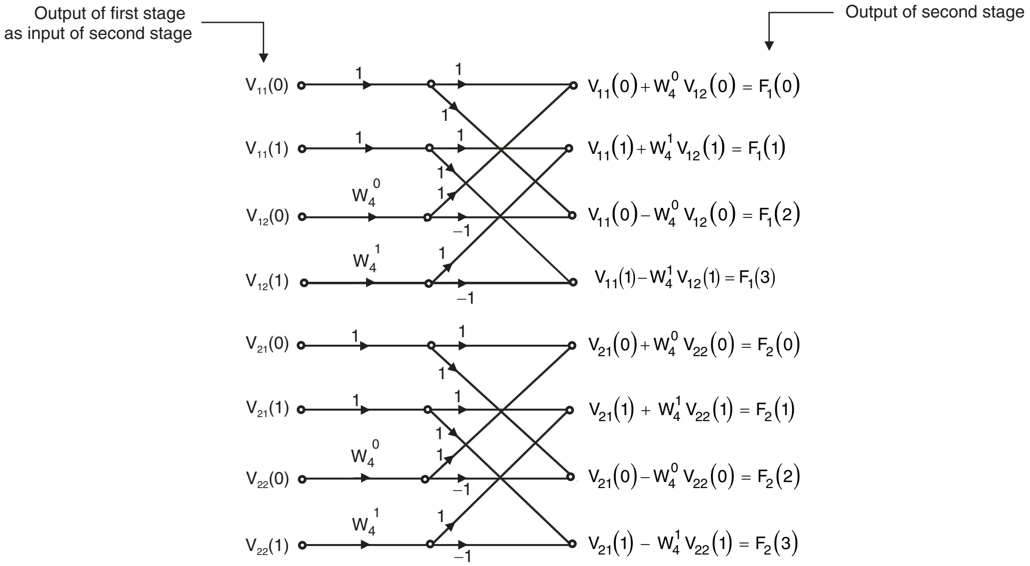


Fig 1.40: *Second stage of flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIT FFT.*

Flow Graph (or Butterfly Diagram) for Third Stage of Computation

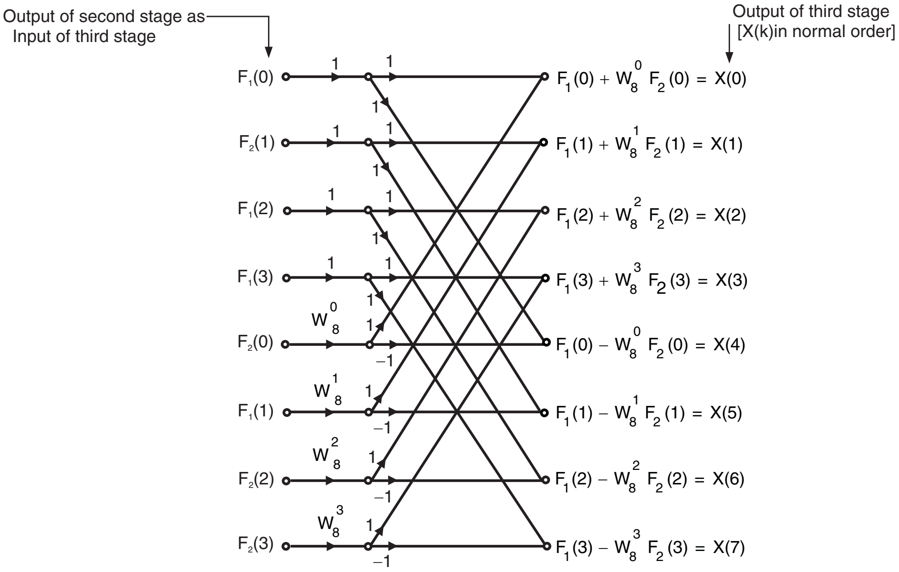


Fig 1.41: *Third stage of flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIT FFT.*

The Combined Flow Graph (or Butterfly Diagram) of All the Three Stages of Computation

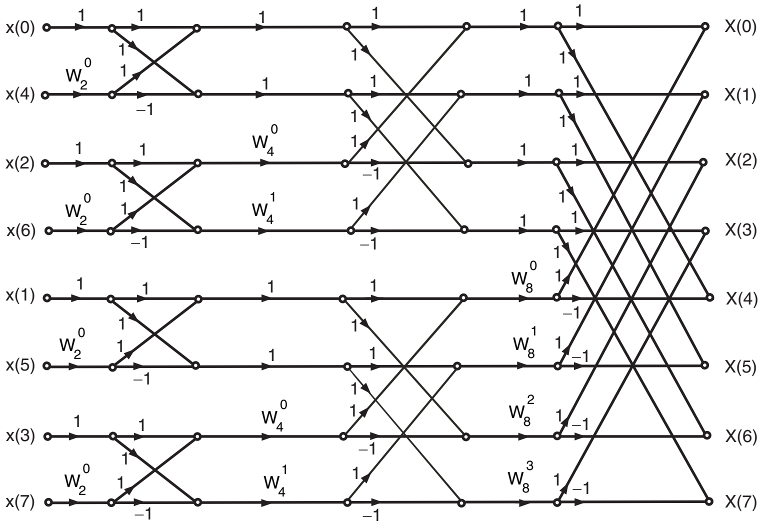


Fig 1.42: The flow graph (or butterfly diagram) for 8-point DIT radix-2 FFT.

1.10 Radix-2 Decimation in Frequency (DIF) Fast Fourier Transform (FFT)

In decimation in frequency algorithm, the frequency domain sequence X(k) is decimated, (but in decimation in time algorithm, the time domain sequence x(n) is decimated).

In this algorithm, the N-point time domain sequence is converted to two numbers of N/2 point sequences. Then each N/2 point sequence is converted to two numbers of N/4 point sequences. Thus, we get 4 numbers of N/4 point sequences.

This process is continued until we get N/2 numbers of 2-point sequences. Finally the 2-point DFT of each 2-point sequence is computed. The 2-point DFTs of N/2 numbers of 2-point sequences will give N samples, which is the N-point DFT of the time domain sequence.

Here the equations for forming N/2 point sequences, N/4 point sequences, etc., are obtained by decimation of frequency domain sequences. Hence, this method is called DIF. For example, the N-point frequency domain sequence X(k) can be decimated to two numbers of N/2 point frequency domain sequences $G_1(k)$ and $G_2(k)$. The $G_1(k)$ and $G_2(k)$ defines new time domain sequences $g_1(n)$ and $g_2(n)$, respectively, whose samples are obtained from x(n).

It can be shown that the N-point DFT of x(n) can be realised from two numbers of N/2 point DFTs. The N/2 point DFTs can be realised from two numbers of N/4 point DFTs and so on. The decimation continues upto 2-point DFTs.

Let x(n) and X(k) be N-point DFT pair.

Let $G_1(k)$ and $G_2(k)$ be two numbers of N/2 point sequences obtained by the decimation of X(k).

Let $G_1(k)$ be N/2 point DFT of $g_1(n)$, and $G_2(k)$ be N/2 point DFT of $g_2(n)$.

Now, the N-point DFT X(k) can be obtained from the two numbers of N/2 point DFTs $G_1(k)$ and $G_2(k)$, as shown below.

$$X(k)\Big|_{k=\text{even}} = G_1(k)$$

$$X(k)\Big|_{k=\text{odd}} = G_2(k)$$

Proof:

By definition of DFT, the N-point DFT of x(n) is,

$$X(k) = \sum_{n=0}^{N-1} x(n)\,W_N^{kn} = \sum_{n=0}^{\frac{N}{2}-1} x(n)\,W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n)\,W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\,W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right)W_N^{k\left(n+\frac{N}{2}\right)} = \sum_{n=0}^{\frac{N}{2}-1} x(n)\,W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right)W_N^{kn}\,W_N^{\frac{kN}{2}}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n)\,W_N^{kn} + (-1)^k\,x\left(n+\frac{N}{2}\right)W_N^{kn}\right] = \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^k\,x\left(n+\frac{N}{2}\right)\right]W_N^{kn} \qquad .....(1.79)$$

$$W_N^{\frac{kN}{2}} = e^{-j\frac{2\pi}{N}\frac{kN}{2}} = e^{-j\pi k} = (e^{-j\pi})^k = (-1)^k$$

Let us split X(k) into even and odd numbered samples.

$$X(k)\Big|_{k=\text{even}} = X(2k) \quad ; \text{ for } k = 0, 1, 2, ..., \frac{N}{2}-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^{2k}x\left(n+\frac{N}{2}\right)\right]W_N^{2kn}$$

Using equation (1.79), replacing k by 2k.

$(-1)^{2k} = 1$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + x\left(n+\frac{N}{2}\right)\right]W_{N/2}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} g_1(n)W_{N/2}^{kn} = G_1(k)$$

$g_1(n) = x(n) + x\left(n+\frac{N}{2}\right)$; for $n = 0, 1, 2, ..., \frac{N}{2}-1$

$G_1(k)$ is $\frac{N}{2}$ point DFT of $g_1(n)$.

$\therefore G_1(k) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n)W_{N/2}^{kn}$; for $k = 0, 1, 2, ..., \frac{N}{2}-1$

$$X(k)\Big|_{k=\text{odd}} = X(2k+1) \quad ; \text{ for } k = 0, 1, 2, ..., \frac{N}{2}-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^{(2k+1)}x\left(n+\frac{N}{2}\right)\right]W_N^{(2k+1)n}$$

Using equation (1.79), replacing k by 2k + 1.

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) - x\left(n+\frac{N}{2}\right)\right]W_N^{2kn}\,W_N^{n}$$

$(-1)^{2k+1} = -1$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) - x\left(n+\frac{N}{2}\right)\right]W_{N/2}^{kn}W_N^{n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} g_2(n)W_{N/2}^{kn} = G_2(k)$$

$g_2(n) = \left(x(n) - x\left(n+\frac{N}{2}\right)\right)W_N^{n}$; for $n = 0, 1, 2, .., \frac{N}{2}-1$

$G_2(k)$ is $\frac{N}{2}$ point DFT of $g_2(n)$.

$\therefore G_2(k) = \sum_{n=0}^{\frac{N}{2}-1} g_2(n)W_{N/2}^{kn}$; for $k = 0, 1, 2, ..., \frac{N}{2}-1$

In the next stage of decimation the N/2 point frequency domain sequence $G_1(k)$ is decimated into two numbers of N/4 point sequences $D_{11}(k)$ and $D_{12}(k)$, and $G_2(k)$ is decimated into two numbers of N/4 point sequences $D_{21}(k)$ and $D_{22}(k)$.

Let, $D_{11}(k)$ and $D_{12}(k)$ be two numbers of N/4 point sequences obtained by the decimation of $G_1(k)$.

Let, $D_{11}(k)$ be N/4 point DFT of $d_{11}(n)$, and $D_{12}(k)$ be N/4 point DFT of $d_{12}(n)$.

Let, $D_{21}(k)$ and $D_{22}(k)$ be two numbers of N/4 point sequences obtained by the decimation of $G_2(k)$.

Let, $D_{21}(k)$ be N/4 point DFT of $d_{21}(n)$, and $D_{22}(k)$ be N/4 point DFT of $d_{22}(n)$.

Now, N/2 point DFTs can be obtained from two numbers of N/4 point DFTs as shown below.

$$G_1(k)\Big|_{k\,=\,\text{even}} = D_{11}(k)$$

$$G_1(k)\Big|_{k\,=\,\text{odd}} = D_{12}(k)$$

$$G_2(k)\Big|_{k\,=\,\text{even}} = D_{21}(k)$$

$$G_2(k)\Big|_{k\,=\,\text{odd}} = D_{22}(k)$$

Proof:

By definition of DFT, the N/2 point DFT of $G_1(k)$ is,

$$W_N^{\frac{kN}{2}} = e^{-j\frac{2\pi}{N}\frac{kN}{2}} = e^{-j\pi k}$$
$$= (e^{-j\pi})^k$$
$$= (-1)^k$$

$$G_1(k) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n)W_{N/2}^{kn} = \sum_{n=0}^{\frac{N}{4}-1} g_1(n)W_{N/2}^{kn} + \sum_{n=\frac{N}{4}}^{\frac{N}{2}-1} g_1(n)W_{N/2}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} g_1(n)W_{N/2}^{kn} + \sum_{n=0}^{\frac{N}{4}-1} g_1\left(n+\frac{N}{4}\right)W_{N/2}^{k\left(n+\frac{N}{4}\right)} = \sum_{n=0}^{\frac{N}{4}-1} g_1(n)W_{N/2}^{kn}$$

$$+ \sum_{n=0}^{\frac{N}{4}-1} g_1\left(n+\frac{N}{4}\right)W_{N/2}^{kn}\,W_{N/2}^{k\frac{N}{4}}$$

$$d_{11}(n) = g_1(n) + g_1\left(n+\frac{N}{4}\right)$$

$D_{11}(k)$ is $\frac{N}{4}$ point DFT of $d_{11}(n)$.

$$\therefore D_{11}(k) = \sum_{n=0}^{\frac{N}{4}-1} d_{11}(n)\,W_{N/4}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + W_{N/2}^{kn/4}\,g_1\left(n+\frac{N}{4}\right)\right]W_{N/2}^{kn}$$

$$d_{12}(n) = \left[g_1(n) - g_1\left(n+\frac{N}{4}\right)\right]W_{N/2}^{n}$$

$D_{12}(k)$ is $\frac{N}{4}$ point DFT of $d_{12}(n)$.

$$\therefore D_{12}(k) = \sum_{n=0}^{\frac{N}{4}-1} d_{12}(n)W_{N/4}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + (-1)^k g_1\left(n+\frac{N}{4}\right)\right]W_{N/2}^{kn} \qquad .....(1.80)$$

Let us split $G_1(k)$ into even and odd numbered samples.

$$G_1(k)\Big|_{k=\text{even}} = G_1(2k) \quad ;\ \text{for } k = 0, 1, 2, ..., \frac{N}{4}-1$$

Using equation (1.80), replacing k by 2k.

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + (-1)^{2k} g_1\left(n+\frac{N}{4}\right)\right]W_{N/2}^{2kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + g_1\left(n+\frac{N}{4}\right)\right]W_{N/4}^{kn} = \sum_{n=0}^{\frac{N}{4}-1} d_{11}(n)W_{N/4}^{kn} = D_{11}(k)$$

$(-1)^{2k} = 1$

$$G_1(k)\Big|_{k=\text{odd}} = G_1(2k+1) \quad ;\ \text{for } k = 0, 1, 2, ..., \frac{N}{4}-1$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + (-1)^{(2k+1)} g_1\left(n+\frac{N}{4}\right)\right]W_{N/2}^{(2k+1)n}$$

Using equation (1.80), replacing k by 2k + 1.

$(-1)^{2k+1} = -1$

$$\therefore G_1(k)\Big|_{k=\text{odd}} = \sum_{n=0}^{\frac{N}{4}-1}\left[g_1(n)-g_1\left(n+\frac{N}{4}\right)\right]W_{N/2}^{n}\,W_{N/4}^{kn} = \sum_{n=0}^{\frac{N}{4}-1} d_{12}(n)W_{N/4}^{kn} = D_{12}(k)$$

Similarly the N/2 point sequence $G_2(k)$ can be decimated into two numbers of N/4 point sequences.

$$G_2(k)\Big|_{k=\text{even}} = G_2(2k) \quad ;\text{ for } k = 0,1,2,\ldots\ldots\ldots,\frac{N}{4}-1$$

$$= \sum_{n=0}^{\frac{N}{4}-1} d_{21}(n)W_{N/4}^{kn} = D_{21}(k)$$

$$G_2(k)\Big|_{k=\text{odd}} = G_2(2k+1) \quad ;\text{ for } k = 0,1,2,\ldots\ldots\ldots,\frac{N}{4}-1$$

$$= \sum_{n=0}^{\frac{N}{4}-1} d_{22}(n)W_{N/4}^{kn} = D_{22}(k)$$

$$d_{22}(n) = \left[g_2(n)-g_2\left(n+\frac{N}{4}\right)\right]W_{N/2}^{n}$$

$D_{22}(k)$ is $\frac{N}{4}$ point DFT of $d_{22}(n)$.

$$\therefore D_{22}(k) = \sum_{n=0}^{\frac{N}{4}-1} d_{22}(n)W_{N/4}^{kn}$$

$$d_{21}(n) = g_2(n)+g_2\left(n+\frac{N}{4}\right)$$

$D_{21}(k)$ is $\frac{N}{4}$ point DFT of $d_{21}(n)$.

$$\therefore D_{21}(k) = \sum_{n=0}^{\frac{N}{4}-1} d_{21}(n)W_{N/4}^{kn}$$

The decimation of the frequency domain sequence can be continued until the resulting sequence are reduced to 2-point sequences. The entire process of decimation involves m stages of decimation where $m = \log_2 N$. The computation of the N-point DFT via the decimation in frequency FFT algorithm requires $(N/2)\log_2 N$ complex multiplications and $N \log_2 N$ complex additions (i.e., the total number of computations remains same in both DIT and DIF).

1.10.1 8- point DFT using Radix-2 DIF FFT

The DIF computations for an eight sequence is discussed in detail in this section. Let x(n) be an 8-point sequence. Therefore, $N = 8 = 2^3 = r^m$. Here, r = 2 and m = 3. Therefore, the computation of 8-point DFT using radix-2 FFT involves three stages of computation.

The samples of x(n) are,

x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)

N = 8

First Stage Computation

In the first stage of computation, two numbers of 4-point sequences $g_1(n)$ and $g_2(n)$ are obtained from x(n) as shown below.

$$g_1(n) = \left[x(n)+x\left(n+\frac{N}{2}\right)\right] = [x(n)+x(n+4)] \;\; ; \text{ for } n = 0,1,2,3$$

When n = 0; $g_1(n) = g_1(0) = x(0) + x(4)$

When n = 1; $g_1(n) = g_1(1) = x(1) + x(5)$

When n = 2; $g_1(n) = g_1(2) = x(2) + x(6)$

When n = 3; $g_1(n) = g_1(3) = x(3) + x(7)$

$$g_2(n) = \left[x(n)-x\left(n+\frac{N}{2}\right)\right]W_N^n = [x(n)-x(n+4)]W_8^n \;\; ; \text{ for } n = 0,1,2,3$$

When n = 0; $g_2(n) = g_2(0) = [x(0) - x(4)]W_8^0$

When n = 1; $g_2(n) = g_2(1) = [x(1) - x(5)]W_8^1$

When n = 2; $g_2(n) = g_2(2) = [x(2) - x(6)]\,W_8^2$

When n = 3; $g_2(n) = g_2(3) = [x(3) - x(7)]\,W_8^3$

Second Stage Computation

In the second stage of computation, 2 numbers of 2-point sequences $d_{11}(n)$ and $d_{12}(n)$ are generated from the samples of $g_1(n)$, and another 2 numbers of 2-point sequences $d_{21}(n)$ and $d_{22}(n)$ are generated from the samples of $g_2(n)$, as shown below.

$$d_{11}(n) = g_1(n) + g_1(n + N/4) = g_1(n) + g_1(n + 2) \;\; ; \text{for } n = 0, 1$$

$$\boxed{N = 8}$$

When n = 0 ; $d_{11}(n) = d_{11}(0) = g_1(0) + g_1(2)$

When n = 1 ; $d_{11}(n) = d_{11}(1) = g_1(1) + g_1(3)$

$$d_{12}(n) = [g_1(n) - g_1(n + N/4)]\,W_{N/2}^{n} = [g_1(n) - g_1(n + 2)]\,W_4^n \;\; ; \text{for } n = 0, 1$$

When n = 0 ; $d_{12}(n) = d_{12}(0) = [g_1(0) - g_1(2)]\,W_4^0$

When n = 1 ; $d_{12}(n) = d_{12}(1) = [g_1(1) - g_1(3)]\,W_4^1$

$$d_{21}(n) = g_2(n) + g_2(n + N/4) = g_2(n) + g_2(n + 2) \;\; ; \text{for } n = 0, 1$$

When n = 0 ; $d_{21}(n) = d_{21}(0) = [g_2(0) + g_2(2)]$

When n = 1 ; $d_{21}(n) = d_{21}(1) = [g_2(1) + g_2(3)]$

$$d_{22}(n) = [g_2(n) - g_2(n + N/4)]\,W_{N/2}^{n} = [g_2(n) - g_2(n + 2)]\,W_4^n \;\; ; \text{for } n = 0, 1$$

When n = 0 ; $d_{22}(n) = d_{22}(0) = [g_2(0) - g_2(2)]\,W_4^0$

When n = 1 ; $d_{22}(n) = d_{22}(1) = [g_2(1) - g_2(3)]\,W_4^1$

Third Stage Computation

In the third stage of computation, 2-point DFTs of the 2-point sequences $d_{11}(n)$, $d_{12}(n)$, $d_{21}(n)$ and $d_{22}(n)$ are computed.

The 2-point DFT of the 2-point sequence $d_{11}(n)$ is computed as shown below.

$$\mathcal{DFT}\{d_{11}(n)\} = D_{11}(k) = \sum_{n=0}^{1} d_{11}(n)\,W_2^{nk} \;\; ; \text{ for } k = 0, 1$$

When k = 0 ; $D_{11}(0) = \sum_{n=0}^{1} d_{11}(n)\,W_2^0 = d_{11}(0) + d_{11}(1)$

$$\boxed{W_2^0 = 1}$$

When k = 1 ; $D_{11}(0) = \sum_{n=0}^{1} d_{11}(n)\,W_2^n = d_{11}(0)\,W_2^0 + d_{11}(1)\,W_2^1$

$$\boxed{W_2^1 = -1 = -1 \times W_2^0}$$

$$= d_{11}(0)\,W_2^0 + d_{11}(1)(-W_2^0) = [d_{11}(0) - d_{11}(1)]\,W_2^0$$

Similarly the 2-point DFTs of the 2-point sequences $d_{12}(n)$, $d_{21}(n)$ and $d_{22}(n)$ are computed and the results are given below.

$$\begin{aligned}
D_{11}(0) &= d_{11}(0) + d_{11}(1) \\
D_{11}(1) &= [d_{11}(0) - d_{11}(1)]\,W_2^0 \\
D_{12}(0) &= d_{12}(0) + d_{12}(1) \\
D_{12}(1) &= [d_{12}(0) - d_{12}(1)]\,W_2^0 \\
D_{21}(0) &= d_{21}(0) + d_{21}(1) \\
D_{21}(1) &= [d_{21}(0) - d_{21}(1)]\,W_2^0 \\
D_{22}(0) &= d_{22}(0) + d_{22}(1) \\
D_{22}(1) &= [d_{22}(0) - d_{22}(1)]\,W_2^0
\end{aligned}$$

Combining the Three Stages of Computation

The final output $D_{ij}(k)$ gives the X(k). The relation can be obtained as shown below.

<table>
<tr><td rowspan="4">$X(2k) = G_1(k)$; k = 0,1,2,3
$\therefore\ X(0) = G_1(0)$
$X(2) = G_1(1)$
$X(4) = G_1(2)$
$X(6) = G_1(3)$</td><td rowspan="4">$X(2k+1) = G_2(k)$; k = 0,1,2,3
$\therefore\ X(1) = G_2(0)$
$X(3) = G_2(1)$
$X(5) = G_2(2)$
$X(7) = G_2(3)$</td><td>$D_{11}(0) = G_1(0) = X(0)$</td></tr>
<tr><td>$D_{11}(1) = G_1(2) = X(4)$</td></tr>
<tr><td>$D_{12}(0) = G_1(1) = X(2)$</td></tr>
<tr><td>$D_{12}(1) = G_1(3) = X(6)$</td></tr>
<tr><td rowspan="2">$G_1(2k) = D_{11}(k)$; k = 0,1
$\therefore\ G_1(0) = D_{11}(0)$
$G_1(2) = D_{11}(1)$</td><td rowspan="2">$G_1(2k+1) = D_{12}(k)$; k = 0,1
$\therefore\ G_1(1) = D_{12}(0)$
$G_1(3) = D_{12}(1)$</td><td>$D_{21}(0) = G_2(0) = X(1)$</td></tr>
<tr><td>$D_{21}(1) = G_2(2) = X(5)$</td></tr>
<tr><td rowspan="2">$G_2(2k) = D_{21}(k)$; k = 0,1
$\therefore\ G_2(0) = D_{21}(0)$
$G_2(2) = D_{21}(1)$</td><td rowspan="2">$G_2(2k+1) = D_{22}(k)$; k = 0,1
$\therefore\ G_2(1) = D_{22}(0)$
$G_2(3) = D_{22}(1)$</td><td>$D_{22}(0) = G_2(1) = X(3)$</td></tr>
<tr><td>$D_{22}(1) = G_2(3) = X(7)$</td></tr>
</table>

From the above we observe that the output is in bit reversed order. In radix-2 DIF FFT, the input is in normal order the output will be in bit reversed order.

1.10.2 Flow Graph For 8-point DFT using Radix-2 DIF FFT

If we observe the basic computation performed at every stage of radix-2 DIF FFT in the previous section, we can arrive at the following conclusion.

1. In each computation two complex numbers "a" and "b" are considered.
2. The sum of the two complex numbers is computed which forms a new complex number "A".

3. Then subtract complex number "b" from "a" to get the term "a-b". The difference term "a-b" is multiplied with the phase factor or twiddle factor "W_N^k" to form a new complex number "B".

The above basic computation can be expressed by a signal flow graph shown in Fig 1.43. (For detailed discussion on signal flow graph, refer Section 1.1.12).

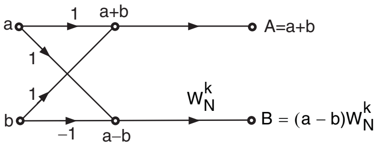


Fig 1.43: *Basic butterfly or flow graph of DIF radix-2 FFT.*

The signal flow graph is also called a ***butterfly diagram*** since it resembles a butterfly. In radix-2 FFT, N/2 butterflies per stage are required to represent the computational process.

The butterfly diagram used to compute the 8-point DFT via radix-2 DIF FFT can be arrived as shown below using the computations shown in previous section.

Flow Graph (or Butterfly Diagram) for First Stage of Computation

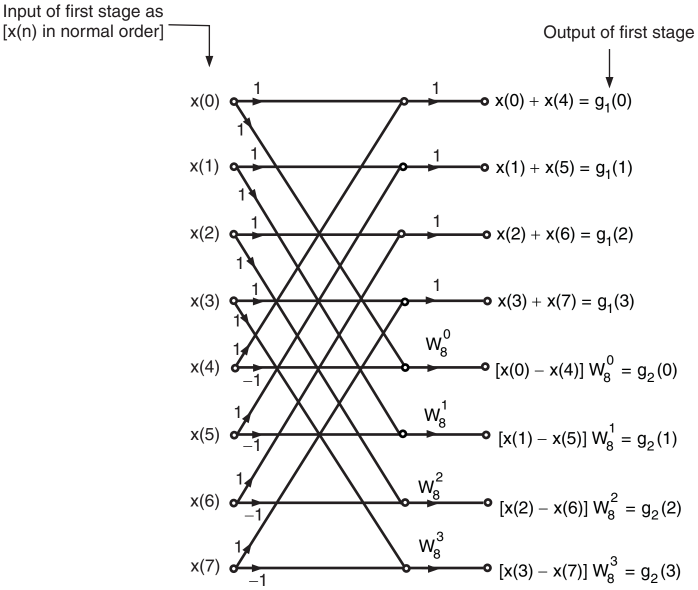


Fig 1.44: *First stage of flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIF FFT.*

Flow Graph (or Butterfly Diagram) for Second Stage of Computation

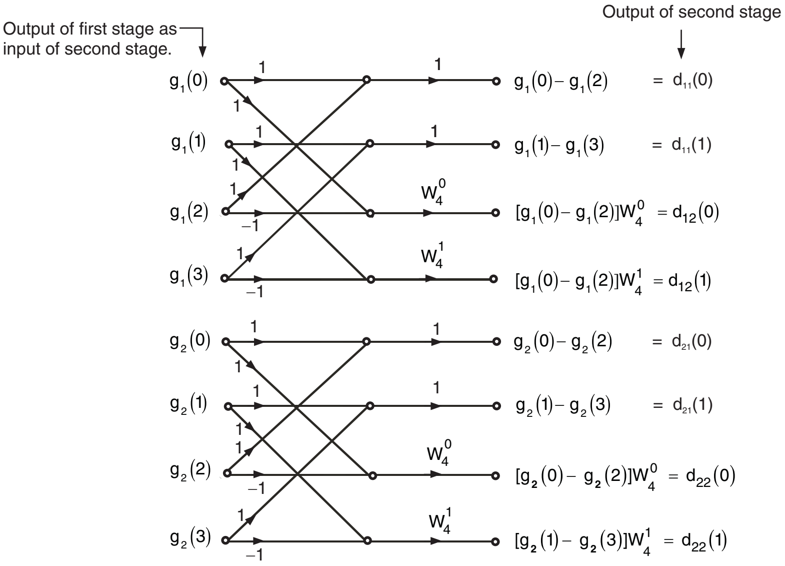


Fig 1.45: *Second stage of flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIF FFT.*

Flow Graph (or Butterfly Diagram) for Third Stage of Computation

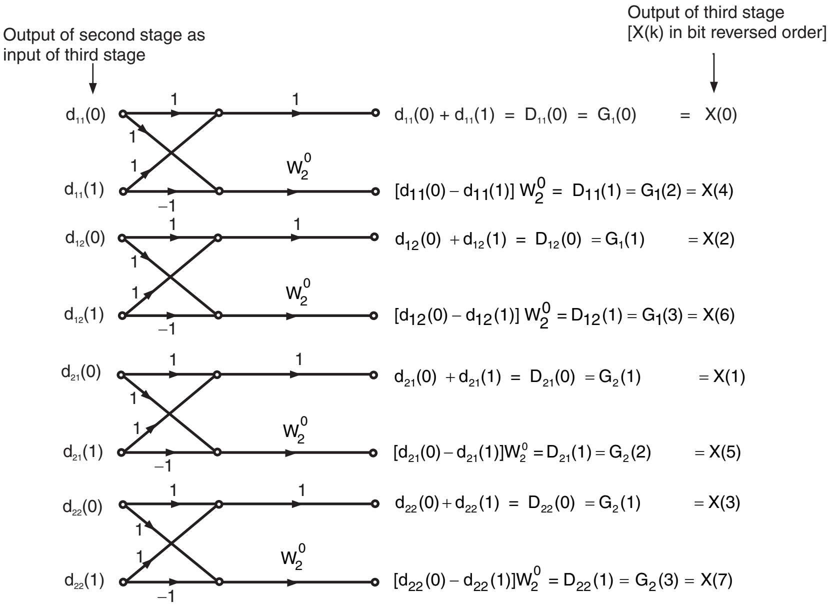


Fig 1.46: *Third stage of flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIF FFT.*

The Combined Flow Graph (or Butterfly Diagram) of All the Three Stages of Computation

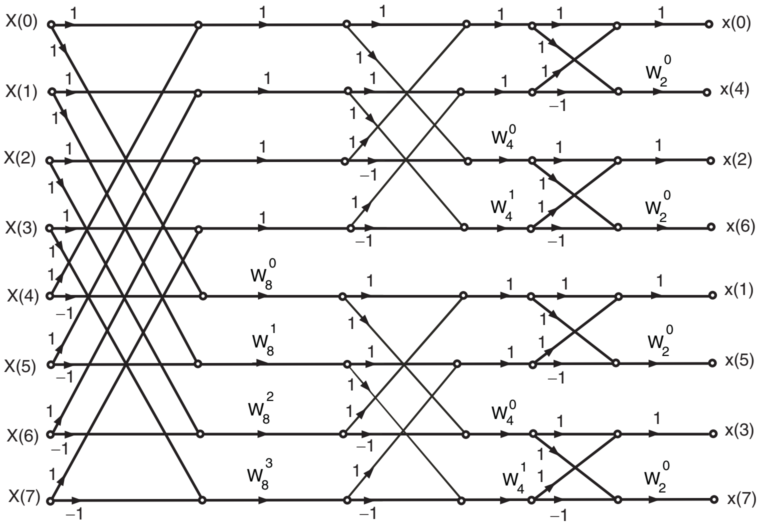


Fig 1.47: *The flow graph (or butterfly diagram) for 8-point DIF via radix-2 FFT.*

1.10.3 Comparison of DIT and DIF Radix-2 FFT

Differences in DIT and DIF

1. In DIT the time domain sequence is decimated, whereas in DIF the frequency domain sequence is decimated.
2. In DIT the input should be in bit-reversed order and the output will be in normal order. For DIF the reverse is true, i.e., the input is in a normal order, while the output is bit reversed.
3. Considering the butterfly diagram, in DIT complex multiplication takes place before the add-subtract operation, whereas in DIF complex multiplication takes place after the add-subtract operation.

Similarities in DIT and DIF

1. For both the algorithms, the value of N should be such that $N = 2^m$ and there will be m stages of butterfly computations with N/2 butterfly per stage.
2. Both the algorithms involve the same number of operations. The total number of complex additions are $N\log_2 N$ and the total number of complex multiplications are $(N/2)\log_2 N$.
3. Both the algorithms require bit reversal at some place during computation.

1.11 Computation of Inverse DFT using FFT

Let, x(n) and X(k) be N-point DFT pair.

Now by the definition of inverse DFT,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi nk}{N}} \quad ; \quad \text{for } n = 0, 1, 2, \text{.......}, N-1$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left(e^{\frac{-j2\pi nk}{N}}\right)^{*} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,(W_N^{nk})^{*} = \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k)\,(W_N^{nk})^{*}\right] \qquad \text{......(1.81)}$$

In equation (1.81), the expression inside the bracket is similar to that of DFT computation of a sequence with following differences.

1. The summation index is k instead of n.
2. The input sequence is X(k) instead of x(n).
3. The phase factors are conjugate of the phase factor used for DFT.

Hence, in order to compute inverse DFT of X(k), the FFT algorithm can be used by taking the conjugate of phase factors. Also from equation (1.81) it is observed that the output of FFT computation should be divided by N to get x(n).

The following procedure can be followed to compute inverse DFT using FFT algorithm.

1. Take N-point frequency domain sequence X(k) as input sequence.
2. Compute FFT by using conjugate of phase factors.
3. Divide the output sequence obtained in FFT computation by N to get the sequence x(n).

Thus, a single FFT algorithm can be used for evaluation of both DFT and inverse DFT.

Example 1.12

An 8-point sequence is given by x(n) = {2, 1, 2, 1, 1, 2, 1, 2}. Compute 8-point DFT of x(n) by a) radix-2 DIT-FFT and b) radix-2 DIF-FFT. Also sketch the magnitude and phase spectrum.

Solution

a) 8-point DFT by Radix-2 DIT-FFT

The given sequence is first arranged in the bit reversed order as shown in Table 1.

Table 1

x(n) Normal order	x(n) Bit reversed order
x(0) = 2	x(0) = 2
x(1) = 1	x(4) = 1
x(2) = 2	x(2) = 2
x(3) = 1	x(6) = 1
x(4) = 1	x(1) = 1
x(5) = 2	x(5) = 2
x(6) = 1	x(3) = 1
x(7) = 2	x(7) = 2

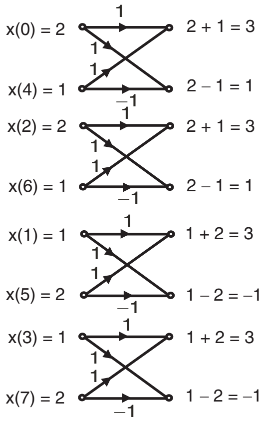


Fig 1: *Butterfly diagram for first stage of radix-2 DIT FFT.*

The phase factor involved in first stage of computation is W_2^0. Since $W_2^0 = 1$, it is not considered for computation.

The 8-point DFT by radix-2 FFT involves three stages of computation with 4-butterfly computations in each stage. The sequence rearranged in the bit reversed order forms the input to the first stage. For other stages of computation, the output of previous stage will be the input for the current stage.

First stage computation

The input sequence of first stage computation = { 2, 1, 2, 1, 1, 2, 1, 2}

The butterfly computations of first stage are shown in Fig 1.

The output sequence of first stage of computation = { 3, 1, 3, 1, 3, −1, 3, −1 }

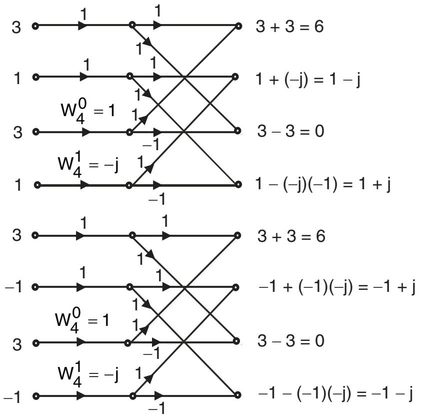
Second stage computation

The input sequence to second stage computation = { 3, 1, 3, 1, 3, –1, 3, –1 }

The phase factors involved in second stage computation are W_4^0 and W_4^1.

The butterfly computations of second stage are shown in Fig 2.

The output sequence of second stage of computation = { 6, 1 – j, 0, 1 + j, 6, –1 + j, 0, –1 – j }



$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = e^0 = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

Fig 2: Butterfly diagram for second stage of radix-2 DIT FFT.

Third stage computation

The input sequence to third stage computation = {6, 1 – j, 0, 1 + j, 6, –1 + j, 0, –1 – j}

The phase factors involved in third stage computation are W_8^0, W_8^1, W_8^2, and W_8^3.

The butterfly computations of third stage are shown in Fig 3.

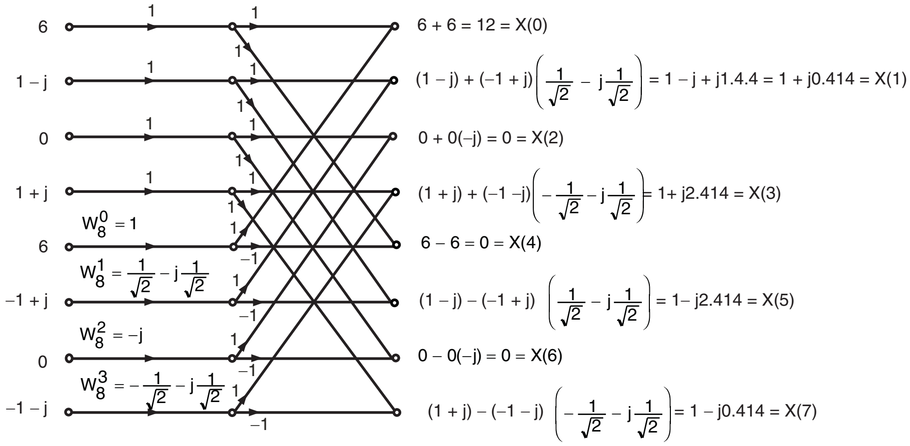


Fig 3: Butterfly diagram for third stage of radix-2 DIT FFT of X(k).

$$W_8^0 = e^{-j2\pi \times \frac{0}{8}} = e^0 = 1$$

$$W_8^1 = e^{-j2\pi \times \frac{1}{8}} = e^{-j \times \frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = e^{-j2\pi \times \frac{2}{8}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi \times \frac{3}{8}} = e^{-j \times \frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

The output sequence of third stage of computation $= \{\, 12,\ 1 + j0.414,\ 0,\ 1 + j2.414,\ 0,\ 1 - j2.414,\ 0,\ 1 - j0.414 \,\}$

The output sequence of third stage of computation is the 8-point DFT of the given sequence in normal order.

$$\boxed{\therefore\ \mathcal{DFT}\{\, x(n) \,\} = X(k) = \{\, 12,\ 1 + j0.414,\ 0,\ 1 + j2.414,\ 0,\ 1 - j2.414,\ 0,\ 1 - j0.414 \,\}}$$

---

b) 8-point DFT by Radix-2 DIF-FFT

For 8-point DFT by radix-2 FFT we require three stages of computation with 4-butterfly computations in each stage. The given sequence is the input to first stage. For other stages of computations, the output of previous stage will be the input for the current stage.

First stage computation

The input sequence for first stage of computation $= \{\, 2,\ 1, 2,\ 1,\ 1,\ 2,\ 1,\ 2 \,\}$

The phase factors involved in first stage computation are $W_8^0, W_8^1, W_8^2,$ and W_8^3.

The butterfly computations of first stage are shown in Fig 4.

The output sequence of third stage of computation $= \{\, 3,\ 3,\ 3,\ 3,\ 1,\ -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}},\ -j\ ,\ \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\ \}$

$$W_8^0 = e^{-j2\pi\times\frac{0}{8}} = e^0 = 1$$

$$W_8^1 = e^{-j2\pi\times\frac{1}{8}} = e^{-j\times\frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = e^{-j2\pi\times\frac{2}{8}} = e^{-j\times\frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi\times\frac{3}{8}} = e^{-j\times\frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

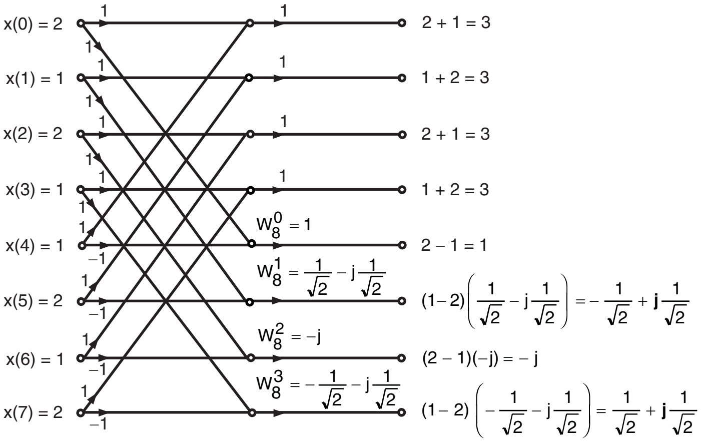


***Fig 4:** Butterfly diagram for first stage of radix-2 DIF FFT.*

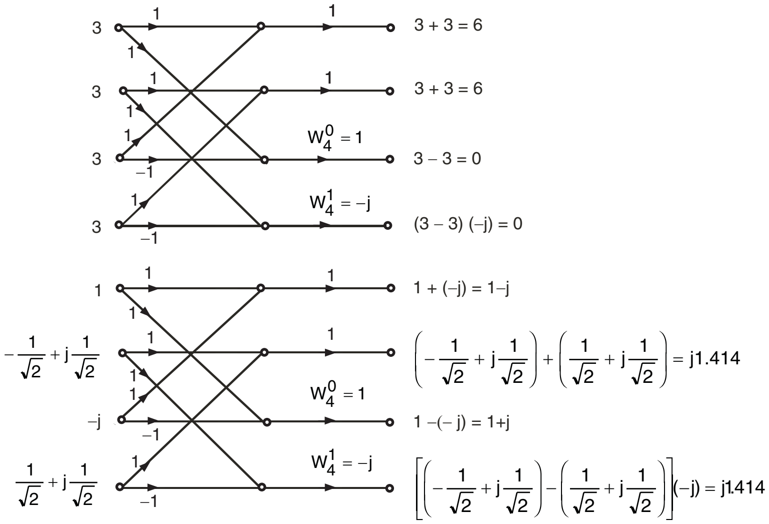
Second stage computation

The input sequence for second stage of computation $= \{\, 3,\ 3,\ 3,\ 3,\ 1,\ -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}},\ -j,\ \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\ \}$

The phase factors involved in second stage computation are W_4^0 and W_4^1 .

The butterfly computations of second stage are shown in Fig 5.

The output sequence of second stage of computation $= \{\, 6,\ 6,\ 0,\ 0\ ,\ 1 - j,\ j1.414,\ 1 + j,\ j1.414 \,\}$



$$W_4^0 = e^{-j2\pi\times\frac{0}{4}} = e^0 = 1$$

$$W_4^1 = e^{-j2\pi\times\frac{1}{4}} = e^{-j\times\frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

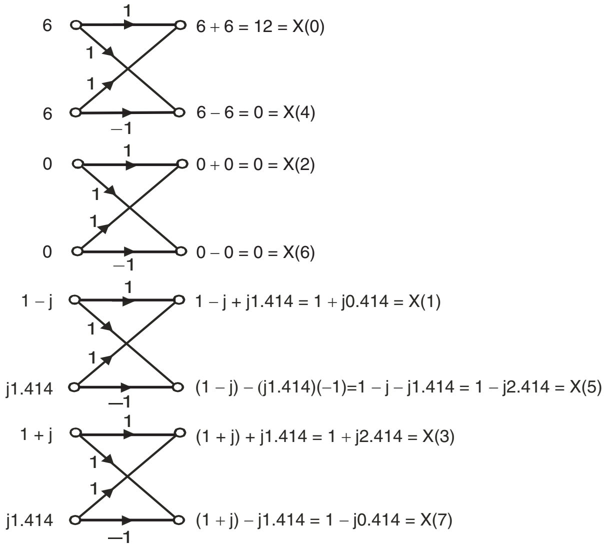
$$= -j$$

Fig 5: *Butterfly diagram for second stage of radix-2 DIF FFT.*

Third stage computation

The input sequence to third stage of computation $= \{6, \ 6, \ 0, \ 0\ , \ 1-j, \ j1.414, \ 1+j, \ j1.414\}$

The butterfly computations of third stage are shown in Fig 6.



The phase factor involved in third stage of computation is W_2^0. Since $W_2^0 = 1$, it is not considered for computation.

Fig 6: *Butterfly diagram for third stage of radix-2 DIF FFT.*

The output sequence of third stage of computation $= \{12, \ 0, \ 0, \ 0, \ 1+j0.414, \ 1-j2.414, \ 1+j2.414, \ 1-j0.414\}$

The output sequence of third stage of computation is the 8-point DFT of the given sequence in bit reversed order.

In DIF-FFT algorithm, the input to the first stage is in normal order and the output of the third stage is in the bit reversed order. Hence, the actual result is obtained by arranging the output sequence of third stage in normal order as shown in Table 2.

Table 2

X(k) Bit reversed order	X(k) Normal order
$X(0) = 12$	$X(0) = 12$
$X(4) = 0$	$X(1) = 1 + j0.414$
$X(2) = 0$	$X(2) = 0$
$X(6) = 0$	$X(3) = 1 + j2.414$
$X(1) = 1 + j0.414$	$X(4) = 0$
$X(5) = 1 - j2.414$	$X(5) = 1 - j2.414$
$X(3) = 1 + j2.414$	$X(6) = 0$
$X(7) = 1 - j0.414$	$X(7) = 1 - j0.414$

$$\boxed{\therefore \mathcal{DFT}\{x(n)\} = X(k) = \{12,\ \ 1+j0.414,\ \ 0,\ \ 1+j2.414,\ \ 0,\ \ 1-j2.414,\ \ 0,\ \ 1-j0.414\}}$$

Magnitude and phase specturm

Each element of the sequence X(k) is a complex number and they are expressed in rectangular coordinates. If they are converted to polar coordinates then the magnitude and phase of each element can be obtained.

> ***Note:*** *The rectangular to polar conversion can be obtained by using R → P conversion in calculator.*

$$X(k) = \{12,\ 1+j0.414,\ 0,\ 1+j2.414,\ 0,\ 1-j2.414,\ 0,\ 1-j0.414\}$$

$$= \{12\angle 0^{\circ},\ 1.08\angle 22^{\circ},\ 0\angle 0^{\circ},\ 2.61\angle 67^{\circ}, 0\angle 0^{\circ},\ 2.61\angle -67^{\circ},\ 0\angle 0^{\circ},\ 1.08\angle -22^{\circ}\}$$

$$= \left\{\begin{matrix} 12\angle 0^{\circ},\ 1.08\angle 22^{\circ}\times\dfrac{\pi}{180^{\circ}},\ 0\angle 0^{\circ},\ 2.61\angle 67^{\circ}\times\dfrac{\pi}{180^{\circ}},\ 0\angle 0^{\circ} \\ 2.61\angle -67^{\circ}\times\dfrac{\pi}{180^{\circ}},\ 0\angle 0^{\circ},\ 1.08\angle -22^{\circ}\times\dfrac{\pi}{180^{\circ}} \end{matrix}\right\}$$

$$= \{12\angle 0,\ 1.08\angle 0.12\pi,\ 0\angle 0,\ 2.61\angle 0.37\pi,\ 0\angle 0,\ 2.61\angle -0.37\pi,\ 0\angle 0,\ 1.08\angle -0.12\pi\}$$

$$\therefore |X(k)| = \{12,\ 1.08,\ 0,\ 2.61,\ 0,\ 2.61,\ 0,\ 1.08\}$$

$$\angle X(k) = \{0,\ 0.12\pi,\ 0,\ 0.37\pi,\ 0,\ -0.37\pi,\ 0,\ -0.12\pi\}$$

The magnitude specturm is the plot of the magnitude of each sample of X(k) as a function of k as shown in Fig 7. The phase spectrum is the plot of phase of X(k) as a function of k as shown in Fig 8.

When N-point DFT is performed on a sequence x(n), then the DFT sequence X(k) will have a periodicity of N. Hence, in this example the magnitude and phase specturm will have a periodicity of 8 as shown in Figs 7 and 8.

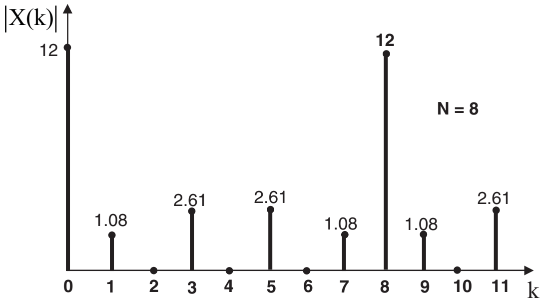


Fig 7 : *Magnitude spectrum.*

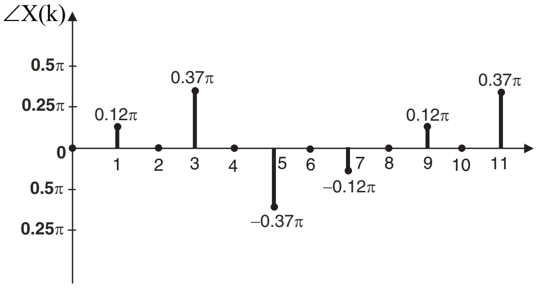


Fig 8: *Phase spectrum.*

1.12 Linear Filtering using FFT

The concept of linear filtering using DFT is discussed in Section 1.7. The drawback in DFT is that the direct computation of DFT involves large number of calculations. This drawback can be overcome by using FFT algorithms for computing the DFT.

The following procedure can be followed to compute response of filter using FFT.

1. Let input x(n) be N_1-point sequence and filter impulse response h(n) be N_2-point sequence and so the response y(n) of the filter is $N_1 + N_2 - 1$ point sequence. Let, $N_1 + N_2 - 1 = M$.
2. Compute M-point FFT of x(n) to get X(k).
3. Compute M-point FFT of h(n) to get H(k).
4. Determine the product of X(k) and H(k). Let, $Y(k) = X(k)\,H(k)$.
5. Take M-point inverse DFT using FFT of Y(k) to get y(n) which is the response of the filter.

***Note:** Refer Examples 1.13 and 1.14 for FIR filter response using FFT.*

Example 1.13

In an FIR filter the input x(n) = {1, 2, 3} and the impulse response of FIR filter h(n) = {–1, –1}. Determine the response of the FIR filter by radix-2 DIT FFT.

Solution

The response y(n) of FIR filter is given by linear convolution of input x(n) and impulse response h(n).

$\therefore$ Response or Output, $y(n) = x(n) * h(n)$

The DFT (or FFT) supports only circular convolution. Hence to get the result of linear convolution from circular convolution, the sequences x(n) and h(n) should be converted to the size of y(n) by appending with zeros and circular convolution of x(n) and h(n) is performed.

The length of x(n) is 3 and h(n) is 2. Hence, the length of y(n) is 3 + 2 – 1 = 4. Therefore, given sequences x(n) and h(n) are converted to 4-point sequences by appending zeros.

$\therefore\ x(n) = \{1, 2, 3, 0\}$ and $h(n) = \{-1, -1, 0, 0\}$

Now the response y(n) is given by, $y(n) = x(n) \circledast h(n)$.

Let, $\mathcal{DFT}\{x(n)\} = X(k)$, $\quad \mathcal{DFT}\{h(n)\} = H(k)$, $\quad \mathcal{DFT}\{y(n)\} = Y(k)$.

By convolution theorem of DFT we get,

$$\mathcal{DFT}\{x(n) \circledast h(n)\} = X(k)\,H(k)$$

$$\therefore\ y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \mathcal{DFT}^{-1}\{X(k)\,H(k)\}$$

The various steps in computing y(n) are,

Step-1 : Determine X(k) using radix-2 DIT algorithm.

Step-2 : Determine H(k) using radix-2 DIT algorithm.

Step-3 : Determine the product X(k)H(k).

Step-4 : Take inverse DFT of the product X(k)H(k) using radix-2 DIT algorithm.

Step-1: To determine X(k)

Since x(n) is a 4-point sequence, we have to compute 4-point DFT. The 4-point DFT by radix-2 FFT consists of two stages of computations with 2-butterflies in each stage. The given sequence x(n) is first arranged in bit reversed order as shown in Table 1.

Table 1

x(n) Normal order	x(n) Bit reversed order
x(0) = 1	x(0) = 1
x(1) = 2	x(2) = 3
x (2) = 3	x(1) = 2
x(3) = 0	x(3) = 0

The sequence arranged in bit reversed order forms the input sequence to first stage computation.

First stage computation

Input sequence to first stage = { 1, 3, 2, 0 }.

The butterfly computations of first stage are shown in Fig 1.

Output sequence of first stage of computation = { 4, – 2, 2, 2 }.

The phase factor involved in first stage of computation is W_2^0. Since $W_2^0 = 1$, it is not considered for computation.

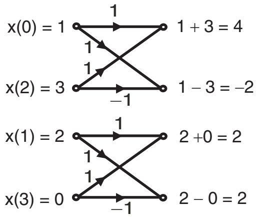


***Fig 1:** Butterfly diagram of first stage of radix-2 DIT FFT of x(n).*

Second stage computation

Input sequence to second stage computation = { 4, – 2, 2, 2 }

The phase factors involved in second stage computation are W_4^0 and W_4^1 .

The butterfly computations of second stage are shown in Fig 2.

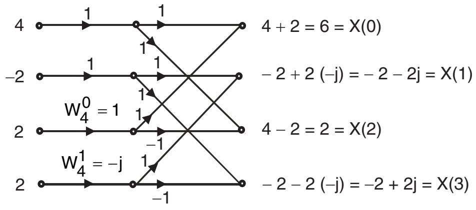


***Fig 2:** Butterfly diagram for second stage of radix-2 DIT FFT of x(n).*

$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = e^0 = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

Output sequence of second stage computation = { 6, –2 – 2j, 2, –2 + 2j }

The output sequence of second stage of computation is the 4-point DFT of x(n).

$$X(k) = \mathcal{DFT}'\{x(n)\} = \{6,\ -2-2j,\ 2,\ -2+2j\}$$

Step-2: To determine H(k)

Since h(n) is a 4-point sequence, we have to compute 4-point DFT. The 4-point DFT by radix-2 FFT consists of two stages of computations with 2-butterflies in each stage. The sequence h(n) is first arranged in bit reversed order as shown in the Table 2.

The sequence in bit reversed order forms the input sequence to first stage computation.

Table 2

h(n) Normal order	h(n) Bit reversed order
h(0) = –1	h(0) = –1
h(1) = –1	h(2) = 0
h(2) = 0	h(1) = –1
h(3) = 0	h(3) = 0

First stage computation

Input sequence of first stage = { –1, 0, –1, 0 }.

The butterfly computations of first stage are shown in Fig 3.

Output sequence of first stage computation = { –1, –1, –1, –1 }

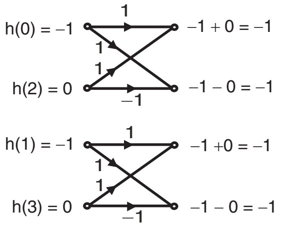


***Fig 3:** Butterfly diagram for first stage of radix-2 DIT FFT of h(n).*

The phase factor involved in first stage of computation is W_2^0. Since $W_2^0 = 1$, it is not considered for computation.

Second stage computation

Input sequence to second stage computation = { −1, −1, −1, −1 }

The phase factors involved are W_4^0 and W_4^1 .

The butterfly computations of second stage are shown in Fig 4.

$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = e^0 = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j\times\frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

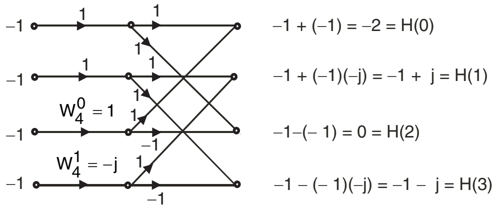


Fig 4: Butterfly diagram for second stage of radix-2 DIT FFT of h(n) .

Output sequence of second stage computation = { −2, −1 + j, 0, −1 − j }

The output sequence of second stage computation is the 4-point DFT of h(n).

$$\boxed{\therefore H(k) = \mathcal{DFT}\{h(n)\} = \{-2,\ -1+j,\ 0,\ -1-j\}}$$

Step-3: To determine the product X(k)H(k)

Let the product, X(k) H(k) = Y(k); for k = 0, 1, 2, 3.

when k = 0; $Y(0) = X(0) \times H(0) = 6 \times (-2) = -12$

when k = 1; $Y(1) = X(1) \times H(1) = (-2-2j) \times (-1+j) = 4$

when k = 2; $Y(2) = X(2) \times H(2) = 2 \times 0 = 0$

when k = 3; $Y(3) = X(3) \times H(3) = (-2+2j) \times (-1-j) = 4$

$$\boxed{\therefore Y(k) = \{-12,\ 4,\ 0,\ 4\}}$$

Step-4: To determine inverse DFT of Y(k)

The 4-point inverse DFT of Y(k) can be computed using radix-2 DIT FFT by taking conjugate of the phase factors and then dividing the output sequence of FFT by 4.

Y(k) = { −12, 4, 0, 4}

The 4-point inverse DFT of Y(k) using radix-2 DIT FFT involves two stages of computations with 2-butterflies in each stage. The sequence Y(k) is arranged in bit reversed order as shown in the Table 3.

The sequence arranged in bit reversed order forms the input sequence to first stage computation.

Table 3

Y(k) Normal order	Y(k) Bit reversed order
Y(0) = −12	Y(0) = −12
Y(1) = 4	Y(2) = 0
Y(2) = 0	Y(1) = 4
Y(3) = 4	Y(3) = 4

First stage computation

Input sequence to first stage = { −12, 0, 4, 4 }.

The butterfly computations of first stage are shown in Fig 5.

The output sequence of first stage computation = { − 12, − 12, 8, 0 }

The phase factor involved in first stage of computation is $(W_2^0)^*$. Since $(W_2^0)^* = 1$, it is not considered for computation.

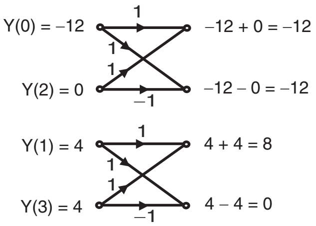


Fig 5: Butterfly diagram for first stage of inverse DFT of Y(k).

Second stage computation

Input sequence to second stage computation = { – 12, – 12, 8, 0 }

The phase factors involved are $(W_4^0)^*$ and $(W_4^1)^*$

$$\boxed{\begin{aligned}(W_4^0)^* &= e^{j2\pi\times\frac{0}{4}} = 1\\ (W_4^0)^* &= e^{j2\pi\times\frac{1}{4}} = e^{j\times\frac{\pi}{2}}\\ &= \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right) = j\end{aligned}}$$

The butterfly computation of second stage is shown in Fig 6.

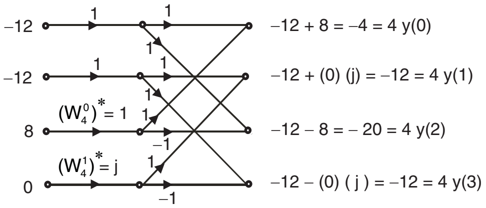


***Fig 6:** Butterfly diagram for second stage of inverse DFT of Y(k).*

The output sequence of second stage computation = { – 4, – 12, – 20, – 12 }

The sequence y(n) is obtained by dividing each sample of output sequence of second stage by 4.

$$\boxed{\therefore \text{ The response of the FIR filter, } y(n) = \left\{-\frac{4}{4},\ -\frac{12}{4},\ -\frac{20}{4},\ -\frac{12}{4}\right\} = \{-1,\ -3,\ -5,\ -3\}}$$

Example 1.14

Determine the response of the FIR filter when the input sequence x(n) = {–1, 2, 2, 2, –1} by radix 2 DIT FFT. The impulse response of the FIR filter is h(n) = {–1, 1, –1, 1}.

Solution:

The response of an FIR filter is given by linear convolution of input x(n) and impulse response h(n).

$\therefore$ Response or Output, $y(n) = x(n) * h(n)$.

The DFT (or FFT) supports only circular convolution. Hence, to get the result of linear convolution from circular convolution, the sequence x(n) and h(n) should be converted to the size of y(n) by appending with zeros, and then circular convolution of x(n) and h(n) is performed.

The length of x(n) = 5, and h(n) = 4. Hence the length of y(n) is 5 + 4 – 1 = 8.

Therefore, x(n) and h(n) are converted into 8-point sequence by appending zeros.

$\therefore$ $x(n) = \{-1,\ 2,\ 2,\ 2,\ -1,\ 0,\ 0,\ 0\}$ and $h(n) = \{-1,\ 1,\ -1,\ 1,\ 0,\ 0,\ 0,\ 0\}$

Now, the response y(n) is given by, $y(n) = x(n) \circledast h(n)$.

Let, $\mathcal{DFT}\{x(n)\} = X(k)$, $\mathcal{DFT}\{h(n)\} = H(k)$, $\mathcal{DFT}\{y(n)\} = Y(k)$.

By convolution theorem of DFT we get,

$$\mathcal{DFT}\{x(n) \circledast h(n)\} = X(k)\,H(k)$$

$$\therefore\ y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \mathcal{DFT}^{-1}\{X(k)\,H(k)\}$$

The various steps in computing y(n) are,

Step-1 : Determine X(k) using radix-2 DIT algorithm.

Step-2 : Determine H(k) using radix-2 DIT algorithm.

Step-3 : Determine the product X(k)H(k).

Step-4 : Take inverse DFT of the product X(k)H(k) using radix-2 DIT algorithm.

Step-1: To determine X(k)

Since x(n) is an 8 point sequence, we have to compute 8-point DFT.

The 8-point DFT by radix-2 FFT algorithm consists of three stages of computations with 4-butterflies in each stage. The given sequence x(n) is arranged in bit reversed order as shown in Table 1.

The sequence arranged in bit-reversed order forms the input sequence to the first stage computation.

Table 1

x(n) Bit reversed order	x(n) Normal order
$x(0) = -1$	$x(0) = -1$
$x(1) = 2$	$x(4) = -1$
$x(2) = 2$	$x(2) = 2$
$x(3) = 2$	$x(6) = 0$
$x(4) = -1$	$x(1) = 2$
$x(5) = 0$	$x(5) = 0$
$x(6) = 0$	$x(3) = 2$
$x(7) = 0$	$x(7) = 0$

First stage computation

Input sequence to first stage $= \{-1,\ -1,\ 2,\ 0,\ 2,\ 0,\ 2,\ 0\}$.

The butterfly computation of first stage is shown in Fig 1.

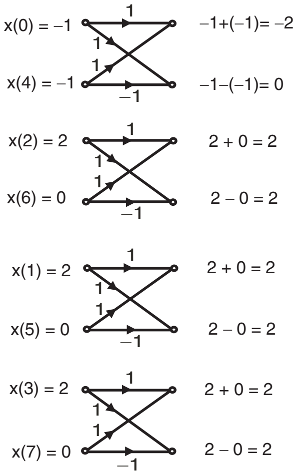


***Fig 1:** Butterfly diagram for first stage of radix-2 DIT FFT of x(n).*

The phase factor involved in first stage of computation is W_2^0. Since $W_2^0 = 1$, it is not considered for computation.

Output sequence of first stage of computation $= \{-2,\ \ 0,\ \ 2,\ \ 2,\ \ 2,\ \ 2,\ \ 2,\ \ 2\}$

Second stage computation

The input sequence to second stage of computation $= \{-2,\ \ 0,\ \ 2,\ \ 2,\ \ 2,\ \ 2,\ \ 2,\ \ 2\}$

Phase factors involved in second stage are W_4^0 and W_4^1.

The butterfly computation of second stage is shown in Fig 2.

Output sequence of second stage of computation $= \{0,\ \ -2j,\ \ -4,\ \ 2j,\ \ 4,\ \ 2-2j,\ \ 0,\ \ 2+2j\}$

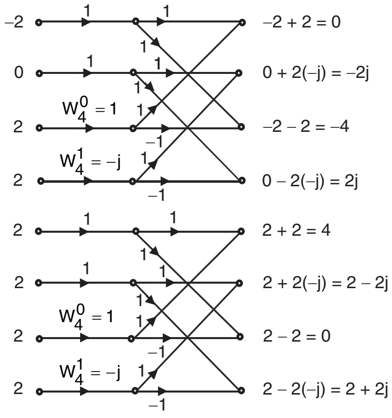


***Fig 2:** Butterfly diagram for second stage of radix-2 DIT FFT of x(n).*

$$W_4^0 = e^{-j2\pi\times\frac{0}{4}} = e^0 = 1$$

$$W_4^1 = e^{-j2\pi\times\frac{1}{4}} = e^{-j\times\frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

Third stage computation

Input sequence to third stage computation = { 0, −2j, −4, 2j, 4, 2 − 2j, 0, 2 + 2j }.

Phase factors involved are W_8^0, W_8^1, W_8^2 and W_8^3.

The butterfly computation of third stage is shown in Fig 3.

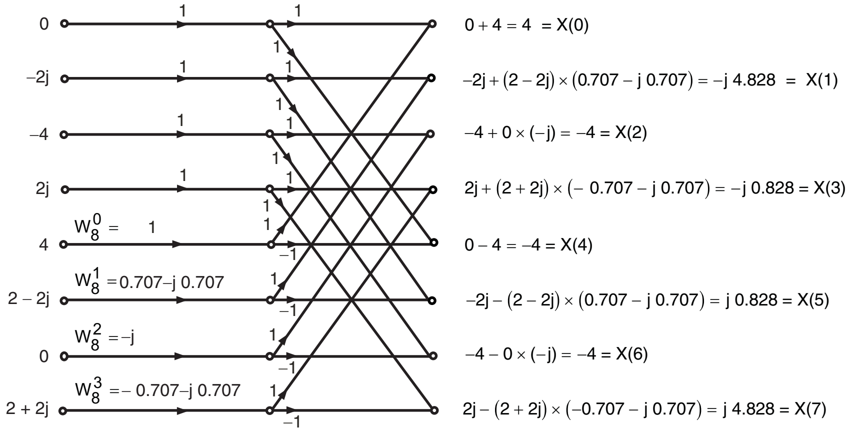


***Fig 3:** Butterfly diagram for third stage of radix-2 DIT FFT of x(n) .*

$$W_8^0 = e^{-j2\pi\times\frac{0}{8}} = e^0 = 1$$

$$W_8^1 = e^{-j2\pi\times\frac{1}{8}} = e^{-j\times\frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = 0.707 - j0.707$$

$$W_8^2 = e^{-j2\pi\times\frac{2}{8}} = e^{-j\times\frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi\times\frac{3}{8}} = e^{-j\times\frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = -0.707 - j0.707$$

The output sequence of third stage of computation = { 4, −j4.828, −4, −j0.828, −4, j0.828, −4, j4.828 }

$$\boxed{\therefore \mathcal{DFT}'\{x(n)\} = X(k) = \{4, -j4.828, -4, -j0.828, -4, j0.828, -4, j4.828\}}$$

Step 2: To determine H(k)

Since h(n) is an 8-point sequence, we have to compute 8-point DFT. The 8-point DFT by radix-2 FFT consists of three stages of computations with 4-butterflies in each stage.

The sequence h(n) is first arranged in bit reversed order as shown in Table 2 .

The sequence arranged in bit reversed order forms the input sequence to the first stage.

Table 2

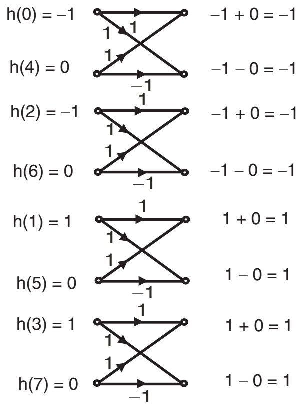
h(n) Normal order	h(n) Bit reversed order
h(0) = −1	h(0) = −1
h(1) = 1	h(4) = 0
h(2) = −1	h(2) = −1
h(3) = 1	h(6) = 0
h(4) = 0	h(1) = 1
h(5) = 0	h(5) = 0
h(6) = 0	h(3) = 1
h(7) = 0	h(7) = 0

First stage computation

Input sequence to first stage computation = { −1, 0, −1, 0, 1, 0, 1, 0 }

The butterfly computations of first stage is shown in Fig 4.

Output sequence of first stage of computation = { −1, −1, −1, −1, 1, 1, 1, 1 }



The phase factor involved in first stage of computation is W_2^0. Since $W_2^0 = 1$, it is not considered for computation.

Fig 4: *Butterfly diagram for first stage of radix-2 DIT FFT of h(n).*

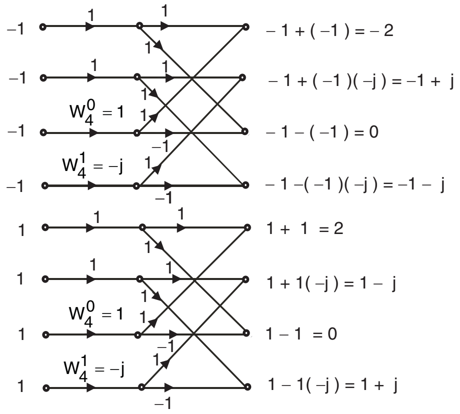
Second stage computation

Input sequence to second stage of computation = { –1, –1, –1, –1, 1, 1, 1, 1 }

Phase factors involved in second stage are W_4^0 and W_4^1.

The butterfly computations of second stage are shown in Fig 5.

Output sequence of second stage of computation = { –2, –1 + j, 0, –1 – j, 2, 1 – j, 0, 1 + j }



$$W_4^0 = e^{-j2\pi\times\frac{0}{4}} = e^0 = 1$$

$$W_4^1 = e^{-j2\pi\times\frac{1}{4}} = e^{-j\times\frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

Fig 5: *Butterfly diagram for second stage of radix-2 DIT FFT of h(n).*

Third stage computation

Input sequence to third stage computation = { –2, –1 + j, 0, –1 – j, 2, 1 – j, 0, 1 + j }

Phase factors involved in third stage computations are W_8^0, W_8^1, W_8^2, and W_8^3.

The butterfly computations of third stage are shown in Fig 6.

$$W_8^0 = e^{-j2\pi\times\frac{0}{8}} = e^0 = 1$$

$$W_8^1 = e^{-j2\pi\times\frac{1}{8}} = e^{-j\times\frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = 0.707 - j0.707$$

$$W_8^2 = e^{-j2\pi\times\frac{2}{8}} = e^{-j\times\frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi\times\frac{3}{8}} = e^{-j\times\frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = -0.707 - j0.707$$

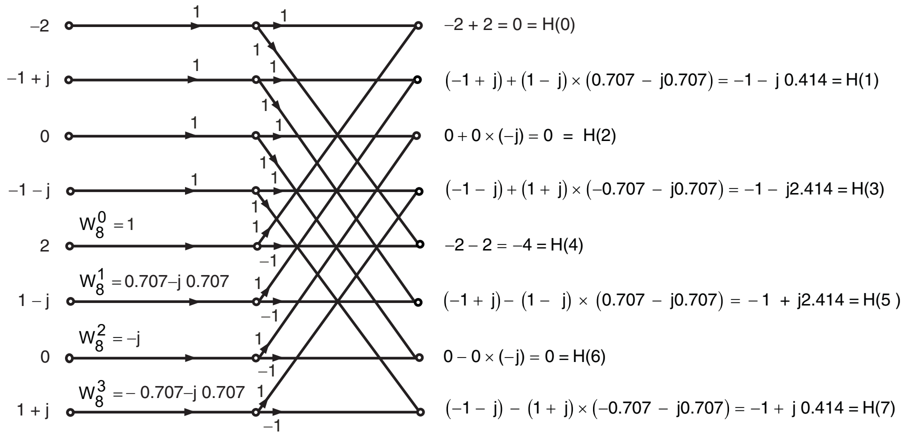


Fig 6: *Butterfly diagram for third stage of radix-2 DIT FFT of h(n) .*

Output sequence to third stage computation = { 0, –1 – j0.414, 0, –1 – j2.414, –4, –1 + j2.414, 0, –1 + j0.414 }

The output sequence of third stage computation is the 8-point DFT of h(n).

$\therefore$ *DFT* { h(n) } = H(k) = { 0, –1 – j0.414, 0, –1 – j2.414, –4, –1 + j2.414, 0, –1 + j0.414 }

Step-3: To determine the product X(k) H(k)

Let the product of X(k)H(k) = Y(k); for k = 0, 1, 2, 3, 4, 5, 6, 7

$\therefore$ Y(k) = X(k)H(k)

When k = 0 ; Y(0) = X(0)H(0) = 4 × 0 = 0

When k = 1 ; Y(1) = X(1)H(1) = –j4.828 × [–1– j0.414] = –2 + j4.828

When k = 2 ; Y(2) = X(2)H(2) = – 4 × 0 = 0

When k = 3 ; Y(3) = X(3)H(3) = –j 0.828 × [–1–j 2.414] = –2 + j 0.828

When k = 4 ; Y(4) = X(4)H(4) = – 4 × –4 = 16

When k = 5 ; Y(5) = X(5)H(5) = j 0.828 × [–1+j 2.414] = –2 – j 0.828

When k = 6 ; Y(6) = X(6)H(6) = –4 × 0 = 0

When k = 7 ; Y(7) = X(7)H(7) = j4.828 × [–1 + j0.414] = – 2 – j4.828

$\therefore$ Y(k) = { 0, –2 + j4.828, 0, –2 + j 0.828, 16, –2 – j 0.828, 0, –2 – j 4.828 }

Step-4: To determine inverse DFT of Y(k)

The 8-point inverse DFT of Y(k) can be computed using radix-2 DIT FFT by taking conjugate of the phase factors and then dividing the output sequence of FFT by 8.

The 8-point inverse DFT of Y(k) using radix-2 DIT FFT involves three stages of computations with 4-butterflies in each stage. The sequence Y(k) is arranged in bit reversed order as shown in Table 3.

The sequence arranged in bit reversed order forms the input sequence to first stage computation.

Table 3

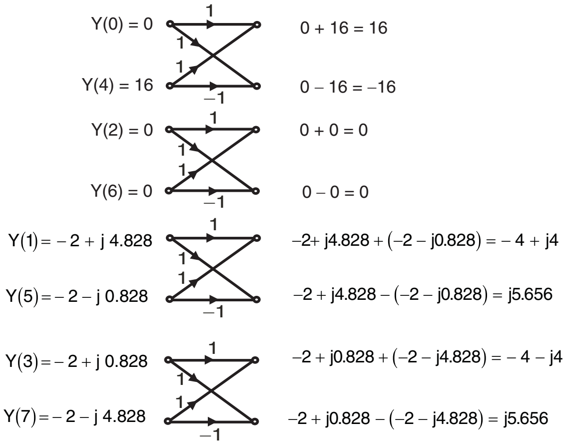
Y(k) Normal order	**Y(k) Bit reversed order**
Y(0) = 0	Y(0) = 0
Y(1) = –2 + j4.828	Y(4) = 16
Y(2) = 0	Y(2) = 0
Y(3) = –2 + j0.828	Y(6) = 0
Y(4) = 16	Y(1) = –2 + j4.828
Y(5) = –2 – j0.828	Y(5) = –2 – j0.828
Y(6) = 0	Y(3) = –2 + j0.828
Y(7) = –2 – j4.828	Y(7) = –2 – j4.828

First stage computation

Input sequence of first stage = { 0, 16, 0, 0 , −2 + j4.828, −2 + j0.828, −2 + j 0.828, −2 − j4.828 }

The butterfly computations of first stage are shown in Fig 7.

Output sequence to first stage = { 16, −16, 0, 0, − 4 + j4, j5.656, − 4 − j4, j5.656 }



The phase factor involved in first stage of computation is $(W_2^0)^*$. Since $(W_2^0)^* = e^{j2\pi\times\frac{0}{4}} = e^0 = 1$, it is not considered for computation.

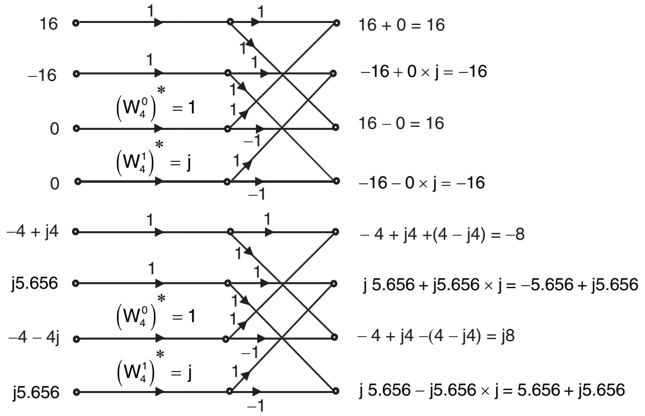
Fig 7: Butterfly diagram for first stage of inverse DFT of Y(k).

Second stage computation

Input sequence of second stage = { 16, −16, 0, 0, −4 + j4, j5.656, −4 − j4, j5.656 }

The butterfly computation of second stage is shown in Fig 8.

The phase factors involved are $(W_4^0)^*$ and $(W_4^1)^*$.



$$(W_4^0)^* = e^{j2\pi\times\frac{0}{4}} = 1$$

$$(W_4^0)^* = e^{j2\pi\times\frac{1}{4}} = e^{j\frac{\pi}{2}}$$

$$= \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right) = j$$

Fig 8: Butterfly diagram for second stage of inverse DFT of Y(k).

Output sequence to second stage = { 16, −16, 16, −16, −8, − 5.656 + j5.656, j8, 5.656 + j5.656 }

Third stage computation

Input sequence of third stage = { 16, −16, 16, −16, − 8, − 5.656 + j 5.656 , j8, + 5.656 + j 5.656 }

The butterfly computation of third stage is shown in Fig 9.

The phase factors involved are $(W_8^0)^*, (W_8^1)^*, (W_8^2)^*$, and $(W_8^3)^*$.

$$(W_8^0)^* = e^{j2\pi\times\frac{0}{8}} = e^0 = 1$$

$$(W_8^1)^* = e^{j2\pi\times\frac{1}{8}} = e^{j\times\frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} = 0.707 + j0.707$$

$$(W_8^2)^* = e^{j2\pi\times\frac{2}{8}} = e^{j\times\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right) = j$$

$$(W_8^3)^* = e^{j2\pi\times\frac{3}{8}} = e^{j\times\frac{3\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) + j\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} = -0.707 + j0.707$$

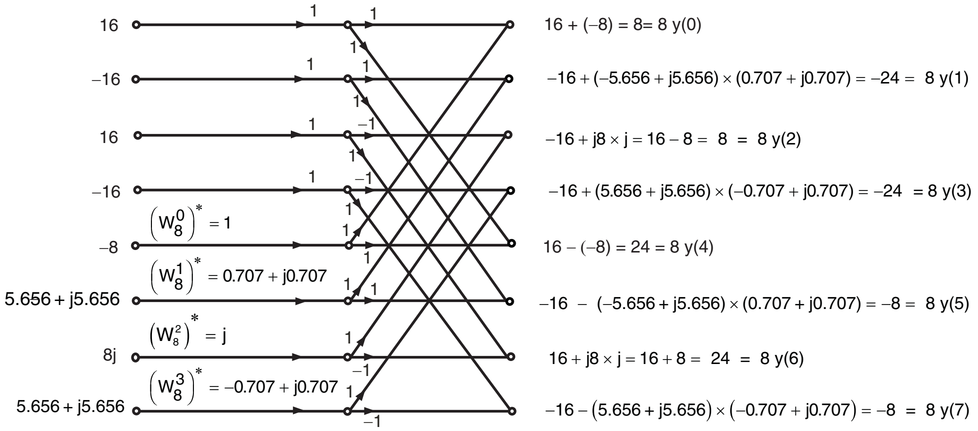


Fig 9: *Butterfly diagram for third stage of inverse DFT of Y(k).*

Output sequence of third stage computation $= \{ 8,\ -24,\ 8,\ -24,\ 24,\ -8,\ 24,\ -8 \}$

The sequence y(n) is obtained by dividing each sample of output sequence of third stage by 8.

$$\therefore \text{ The response of the FIR system, } y(n) = \left\{\frac{8}{8}, \frac{-24}{8}, \frac{8}{8}, \frac{-24}{8}, \frac{-24}{8}, \frac{-8}{8}, \frac{24}{8}, \frac{-8}{8}\right\}$$

$$= \{ 1,\ -3,\ 1,\ -3,\ 3,\ -1,\ 3,\ -1 \}$$

1.13 Summary of Important Concepts

1. A discrete signal is a function of a discrete independent variable.
2. In a discrete time signal, the value of discrete time signal and the independent variable time are discrete.
3. A digital signal is same as a discrete signal except that the magnitude of the signal is quantized.
4. A discrete time sinusoid is periodic only if its frequency is a rational number.
5. Discrete time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.
6. Sampling is the process of a conversion of a continuous time signal into a discrete time signal.
7. The time interval between successive samples is called sampling time or sampling period.
8. The inverse of sampling period is called sampling frequency.
9. Signals that can be completely specified by mathematical equations are called deterministic signals.
10. The convolution of N_1 and N_2 sample sequences produces a sequence consisting of N_1+N_2-1 samples.
11. In an LTI system response for an arbitrary input is given by convolution of input with impulse response.
12. The output sequence of circular convolution is also periodic sequence with periodicity of N samples.
13. The DFT has been developed to convert a continuous function of ω to a discrete function of ω.
14. The DFT of a discrete time signal can be obtained by sampling the DTFT of the signal.

15. The sampling of the DTFT is conventionally performed at N equally spaced frequency points in the period, $0 \le \omega \le 2\pi$.
16. DFT sequence starts at k = 0, corresponding to ω = 0 but does not include k = N, corresponding to ω = 2π.
17. X(k) is also called discrete frequency spectrum (or signal spectrum) of the discrete time signal x(n).
18. The plot of samples of magnitude sequence versus k is called magnitude spectrum.
19. The plot of samples of phase sequence versus k is called phase spectrum.
20. The DFT sequence X(k) is periodic with periodicity of N samples.
21. The DFT of circular convolution of two sequences is equivalent to the product of their individual DFTs.
22. The N-point DFT of a finite duration sequence can be obtained from the Ƶ-transform of the sequence by evaluating the Ƶ-transform at N equally spaced points around the unit circle.
23. The DFT supports only circular convolution and so linear convolution using DFT has to be computed via circular convolution.
24. FFT is a method (or algorithm) for computing the DFT with reduced number of calculations.
25. In N-point DFT by radix-r FFT, the number of stages of computation will be "m" times, where $m = \log_r N$.
26. In direct computation of N-point DFT, the total number of complex additions are N(N–1) and total number of complex multiplications are N^2.
27. In computation of N-point DFT via radix-2 FFT, the total number of complex additions are $N\log_2 N$ and total number of complex multiplications are $(N/2)\log_2 N$.
28. The complex valued phase factor or twiddle factor W_N is defined as, $W_N = e^{-\frac{j2\pi}{N}}$.
29. The term W in phase factor represents a complex number $1\angle{-2\pi}$.
30. In DIT the time domain sequence is decimated, whereas in DIF the frequency domain sequence is decimated.
31. In radix-2 FFT algorithm, the N-point DFT can be realised from two numbers of N/2 point DFTs, the N/2 point DFT can be realised from two numbers of N/4 points DFTs, and so on.
32. In radix-2 FFT, N/2 butterflies per stage are required to represent the computational process.
33. In radix-2 DIT FFT, the input should be in bit reversed order and the output will be in normal order.
34. In radix-2 DIF FFT, the input should be in normal order and the output will be in bit reversed order.
35. In butterfly computation of DIT, the multiplication of phase factor takes place before the add-subtract operation.
36. In butterfly computation of DIF, the multiplication of phase factor takes place after the add-subtract operation.
37. In FFT, the phase factor for computing inverse DFT will be conjugate of phase factors for computing DFT.

1.14 Short-Answer Questions

Q1.1 Express the discrete time signal x(n) as a summation of impulses.

If we multiply a signal x(n) by a delayed unit impulse δ(n – m), then the product is x(m), where x(m) is the signal sample at n = m [because δ(n – m) is 1 only at n = m and zero for other values of n]. Therefore, if we repeat this multiplication over all possible delays in the range $-\infty < m < \infty$ and sum all the product sequences, then the result will be a sequence that is equal to the sequence x(n).

$$\therefore\ x(n) = \ldots x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + \ldots$$

$$= \sum_{m=-\infty}^{\infty} x(m)\,\delta(n-m)$$

Q1.2 State the need for sampling.

Sampling is needed for processing of a continous time signal using its sampled version of signal in digital systems.

Q1.3 ***What are the basic elements used to construct the block diagram of a discrete time system?***

The basic elements used to construct the block diagram of a discrete time system are adder, constant multiplier and unit delay element.

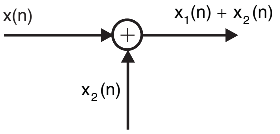


Fig a: *Adder.*

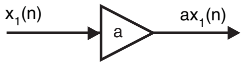


Fig b: *Constant multiplier.*

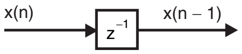
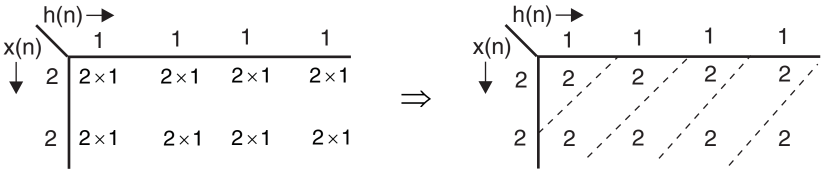


Fig c: *Unit delay element.*

Q1.4 ***Perform convolution of the two sequences, x(n) = {2, 2} and h(n) = {1, 1, 1, 1}***

Solution:

The input sequences x(n) is arranged as a column and the impulse response is arranged as a row as shown below. The elements of the two-dimensional array are obtained by multiplying the corresponding row element with the column element. The sum of the diagonal elements gives the samples of y(n).



The input sequence starts at n = 0 and the impulse response sequence starts at n = 0. Therefore, the output sequence starts at n = 0 + 0 = 0.

The input consits of 4 samples and impulse response consists of 2 samples, so the output consists of 4 + 2 – 1 = 5 samples.

$\therefore$ y(0) = 2

y(1) = 2 + 2 = 4

y(2) = 2 + 2 = 4

y(3) = 2 + 2 = 4

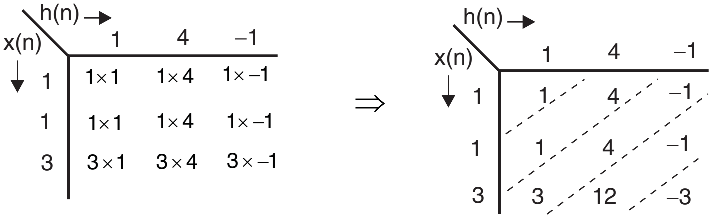
y(4) = 2

$$\therefore y(n) = \{\underset{\uparrow}{2}, 4, 4, 4, 2\}$$

Q1.5 ***Perform convolution of the two sequences x(n) = {1, 1, 3} and h(n) = {1, 4, –1}***

Solution:

The input sequence x(n) is arranged as a column and the impulse response is arranged as a row as shown below. The elements of the two-dimensional array are obtained by multiplying the corresponding row element with the column element. The sum of the diagonal elements gives the samples of y(n).



The input sequence starts at n = 0 and the impulse response sequence starts at n = 0. Therefore, the output sequence starts at n = 0 + 0 = 0.

The input and impulse response consists of 3 samples, so the output consists of 3 + 3 – 1 = 5

$$\therefore \quad \left.\begin{array}{l} y(0) = 1 \\ y(1) = 1 + 4 = 5 \\ y(2) = 3 + 4 - 1 = 6 \end{array}\right| \begin{array}{l} y(3) = 12 - 1 = 11 \\ y(4) = -3 \\ \end{array}$$

$$\therefore\ y(n) = \{\underset{\uparrow}{1}, 5, 6, 11, -3\}$$

Q1.6 *Perform the following convolutions.*

a) $x(n) * \delta(n)$ ***b)*** $\delta(n) * [h_1(n) + h_2(n)]$

Solution:

$$\boxed{\begin{aligned} \delta(n-m) &= 1\ ;\ m = n \\ &= 0\ ;\ m \neq n \end{aligned}}$$

a) $$x(n) * \delta(n) = \sum_{m=-\infty}^{+\infty} x(n)\,\delta(n-m)$$

$$= x(m)\,\delta(n-m)\Big|_{m=n} = x(n)$$

$\boxed{\text{Using solution of (a)}}$

b) $$\delta(n) * [h_1(n) + h_2(n)] = \delta(n) * h_1(n) + \delta(n) * h_2(n)$$

$$= h_1(n) + h_2(n)$$

Q1.7 *Perform the circular convolution of the two sequences* $x_1(n) = \{1, 2, 3\}$ ***and*** $x_2(n) = \{4, 5, 6\}$.

Solution:

Let $x_3(n)$ be the sequence obtained from circular convolution of $x_1(n)$ and $x_2(n)$. The sequence $x_1(n)$ can be arranged as a column vector of order 3 ×1 and using the samples of $x_2(n)$ a 3 × 3 matrix is formed as shown below. The product of two matrices gives the sequence $x_3(n)$.

$$\begin{bmatrix} x_2(0) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 4 & 6 & 5 \\ 5 & 4 & 6 \\ 6 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4\times1 + 6\times2 + 5\times3 \\ 5\times1 + 4\times2 + 6\times3 \\ 6\times1 + 5\times2 + 4\times3 \end{bmatrix} = \begin{bmatrix} 31 \\ 31 \\ 28 \end{bmatrix}$$

$$\therefore\ x_3(n) = x_1(n) \circledast x_2(n) = \{31, 31, 28\}.$$

Q1.8 *Perform the linear convolution of the two sequences* $x_1(n) = \{1, 2\}$ ***and*** $x_2(n) = \{3, 4\}$ ***via circular convolution.***

Solution:

Let $x_3(n)$ be the sequence obtained from linear convolution of $x_1(n)$ and $x_2(n)$. The length of $x_3(n)$ will be 2 + 2 – 1 = 3. Let us convert $x_1(n)$ and $x_2(n)$ into three sample sequences by padding with zeros as shown below.

$x_1(n) = \{1, 2, 0\}$ and $x_2(n) = \{3, 4, 0\}$

Now the circular convolution of $x_1(n)$ and $x_2(n)$ will give $x_3(n)$. The sequence $x_1(n)$ is arranged as a column vector and using the sequence $x_2(n)$, a 3 ×3 matrix is formed as shown below. The product of the two matrices gives the sequence $x_3(n)$.

$$\begin{bmatrix} x_2(0) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 3 & 0 & 4 \\ 4 & 3 & 0 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3\times1 + 0\times2 + 4\times0 \\ 4\times1 + 3\times2 + 0\times0 \\ 0\times1 + 4\times2 + 3\times0 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 8 \end{bmatrix}$$

$$\therefore\ x_3(n) = x_1(n) * x_2(n) = \{3, 10, 8\}$$

Q1.9 *In what way is zero padding implemented in the overlap save method?*

In the overlap save method, zero padding is employed to convert the smaller input sequence to the size of the output sequence of each sectioned convolution. It is also employed to convert either the last section or the first section of the longer input sequence to the size of the output sequence of each sectioned convolution. (This depends on the method of overlapping input samples).

Q1.10 Compare the overlap add and overlap save method of sectioned convolutions.

Overlap add method	**Overlap save method**
1. Linear convolution of each section of longer sequence with smaller sequence is performed.	1. Circular convolution of each section of longer sequence with smaller sequence is performed.(after converting them to the size of the output sequence).
2. Zero padding is not required.	2. Zero padding is required to convert the smaller input sequence to the size of the output sequence.
3. Overlapping of samples of input sections is not required.	3. The N_2-1 samples of an input section of the longer sequence is overlapped with the next input section.
4. The overlapped samples in the output of sectioned convolutions are added to get the overall output.	4. Depending on the method of overlapping the input samples, either last N_2-1 samples or first N_2-1 samples of output sequence of each sectioned convolution are discarded.

Q1.11 Calculate the DFT of the sequence x(n) = {1, 1, –2, –2}.

Solution:

The N-point DFT of x(n) is given by,

$$\mathcal{DFT}\{x(n)\} \;=\; X(k) \;=\; \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi k}{N}} \quad ;\text{for } \; k = 0, 1, 2, ..., N-1$$

Since x(n) is a 4-point sequence, we can take 4-point DFT.

$$\boxed{\begin{matrix} x(0) = \;1, & x(1) = 1 \\ x(2) = -2, & x(3) = -2 \end{matrix}}$$

$$\therefore\; X(k) \;=\; \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi kn}{4}} \;=\; x(0)\, e^{0} + x(1)\, e^{\frac{-j\pi k}{2}} + x(2)\, e^{-j\pi k} + x(3)\, e^{\frac{-j3\pi k}{2}}$$

$$= 1 + e^{\frac{-j\pi k}{2}} - 2e^{-j\pi k} - 2\, e^{\frac{-j3\pi k}{2}} \quad ; \;\text{for } k = 0, 1, 2, 3$$

Q1.12 Find the DFT of the sequence x(n) = {1, 1, 0, 0}. Also find magnitude and phase sequence.

Solution:

The N-point DFT of x(n) is given by,

$$\mathcal{DFT}\{x(n)\} \;=\; X(k) \;=\; \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ;\text{for } \; k = 0, 1, 2, ..., N-1$$

Since x(n) is a 4-point sequence, we can take 4-point DFT.

$$\therefore\; X(k) \;=\; \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi kn}{4}} \;=\; x(0)\, e^{0} + x(1)\, e^{\frac{-j\pi k}{2}} + x(2)\, e^{-j\pi k} + x(3)\, e^{\frac{-j3\pi k}{2}}$$

$$\boxed{e^{j\theta}\, e^{-j\theta} = 1}$$

$$= \; 1 + e^{\frac{-j\pi k}{2}} + 0 + 0 \; = 1 + e^{-\frac{j\pi k}{4}}\, e^{-\frac{j\pi k}{4}} = e^{-\frac{j\pi k}{4}}\left(e^{\frac{j\pi k}{4}} + e^{-\frac{j\pi k}{4}}\right)$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \; e^{-\frac{j\pi k}{4}}\, 2\cos\left(\frac{\pi k}{4}\right) \;=\; 2\cos\left(\frac{\pi k}{4}\right) e^{-\frac{j\pi k}{4}} \quad ;\text{for } k = 0, 1, 2, 3$$

$$\therefore\; \left|X(k)\right| \;=\; 2\cos\left(\frac{\pi k}{4}\right) \text{and } \angle X(k) \;=\; -\frac{\pi k}{4} \quad ; \;\text{for } k = 0, 1, 2, 3$$

Q1.13 Compute the DFT of the sequence x(n) = (–1)ⁿ for the period N = 16.

Solution:

Given that, $x(n) = (-1)^n = \{...... 1, -1, 1, -1, 1, -1, ...\}$.

On evaluating the sequence for all values of n, it can be observed that x(n) is periodic with periodicity of 2 samples. The DFT of x(nhas to be computed for the period N = 16. Let us consider the 16-sample of the infinite sequence from n = 0 to n = 15.

The 16-point DFT of x(n) is given by,

$$X(k) = \sum_{n=0}^{15} x(n)\, e^{-j\frac{2\pi kn}{16}} = \sum_{n=0}^{15} (-1)^n \times e^{-j\frac{\pi kn}{8}} = \sum_{n=0}^{15}\left(-e^{-j\frac{\pi k}{8}}\right)^n$$

$$= \frac{1-\left(-e^{-\frac{j\pi k}{8}}\right)^{16}}{1-\left(-e^{-\frac{j\pi k}{8}}\right)} = \frac{1-e^{-\frac{j\pi k16}{8}}}{1+e^{-\frac{j\pi k}{8}}} = \frac{1-e^{-j2\pi k}}{1+e^{-\frac{j\pi k}{16}}\, e^{-\frac{j\pi k}{16}}}$$

$$= \frac{1-(\cos 2\pi k - j\sin 2\pi k)}{e^{-\frac{j\pi k}{16}}\left(e^{\frac{j\pi k}{16}} + e^{-\frac{j\pi k}{16}}\right)} = \frac{1-\cos 2\pi k}{e^{-\frac{j\pi k}{16}}\, 2\cos\frac{\pi k}{16}}$$

$$= \frac{1-\cos 2\pi k}{2\cos\frac{\pi k}{16}}\, e^{j\frac{\pi k}{16}} \quad ;\text{for } k = 0, 1, 2, 3, ...15$$

$$\boxed{\sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C}}$$

$$\boxed{e^{j\theta}\, e^{-j\theta} = 1}$$

$$\boxed{e^{-j\theta} = \cos\theta - j\sin\theta}$$

$$\boxed{\begin{array}{l}\text{For interger k,}\\ \sin 2\pi k = 0\end{array}}$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

Q1.14 Find the inverse DFT of Y(k) = {1, 0, 1, 0}.

Solution:

The inverse DFT of the sequence Y(k) of length 4 is given by,

$$\mathcal{DFT}^{-1}\{Y(k)\} = y(n) = \frac{1}{4}\sum_{k=0}^{3} Y(k)\, e^{\frac{j2\pi kn}{4}} \quad ;\text{for } n = 0, 1, 2, 3$$

$$\therefore\; y(n) = \frac{1}{4}\left[Y(0)e^0 + Y(1)\, e^{\frac{j\pi n}{2}} + Y(2)\, e^{j\pi n} + Y(3)\, e^{\frac{j3\pi n}{2}}\right]$$

$$= \frac{1}{4}\left[1 + 0 + e^{j\pi n} + 0\right] = \frac{1}{4}\left[1 + \cos\pi n + j\sin\pi n\right] = 0.25(1+\cos\pi n) \;\; ;\text{for } n = 0, 1, 2, 3$$

$$\boxed{\begin{array}{l}\text{For interger n,}\\ \sin \pi n = 0.\end{array}}$$

$$\boxed{\begin{array}{ll} Y(0) = 1, & Y(1) = 0 \\ Y(2) = 1, & Y(3) = 0 \end{array}}$$

When n = 0; $y(0) = 0.25\,(1 + \cos 0) = 0.5$

When n = 1; $y(1) = 0.25\,(1 + \cos \pi) = 0$

When n = 2; $y(2) = 0.25\,(1 + \cos 2\pi) = 0.5$

When n = 3; $y(3) = 0.25\,(1 + \cos 3\pi) = 0$

$\therefore\; y(n) = \{0.5, 0, 0.5, 0\}$

Q1.15 Calculate the percentage saving in calculations in a 512-point radix-2 FFT, when compared to direct DFT.

Solution:

Direct computation of DFT

Number of complex additions = $N(N-1) = 512 \times (512 - 1) = 2{,}61{,}632$

Number of complex multiplications = $N^2 = 512^2 = 2{,}62{,}144$

Radix-2 FFT

Number of complex additions = $N\log_2 N = 512 \times \log_2 512$

$= 512 \times \log_2 2^9 = 512 \times 9 = 4{,}608$

$$\text{Number of complex multiplications} = \frac{N}{2}\log_2 N = \frac{512}{2}\times\log_2 512$$

$$= \frac{512}{2}\times\log_2 2^9 = \frac{512}{2}\times 9 = 2,304$$

Percentage Saving

$$\text{Percentage Saving in additions} = 100 - \frac{\text{Number of additions in radix}-2\text{ FFT}}{\text{Number of additions in direct DFT}}\times 100$$

$$= 100 - \frac{4,608}{2,61,632}\times 100 = 98.2\%$$

$$\text{Percentage Saving in multiplications} = 100 - \frac{\text{Number of multiplications in radix}-2\text{ FFT}}{\text{Number of multiplications in direct DFT}}\times 100$$

$$= 100 - \frac{2,304}{2,62,144}\times 100 = 99.1\%$$

Q1.16 Arrange the 8-point sequence x(n) = {1, 2, 3, 4, –1, –2, –3, –4} in bit reversed order.

The x(n) in normal order = {1, 2, 3, 4, –1, –2, –3, –4}

The x(n) in bit reversed order = {1, –1, 3, –3, 2, –2, 4, –4}

Q1.17 Compare the DIT and DIF radix-2 FFT.

DIT radix-2 FFT	DIF radix-2 FFT
1. The time domain sequence is decimated.	1. The frequency domain sequence is decimated.
2. The input should be in bit reversed order, the output will be in normal order.	2. The input should be in normal order, the output will be in bit reversed order.
3. In each stage of computations the phase factors are multiplied before add and subtract operations.	3. In each stage of computations, the phase factors are multiplied after add and subtract operations.
4. The value of N should be expressed such that $N = 2^m$ and this algorithm consists of m stages of computations.	4. The value of N should be expressed such that, $N = 2^m$ and this algorithm consists of m stages of computations.
5. Total number of arithmetic operations are $N\log_2 N$ complex additions and $(N/2)\log_2 N$ complex multiplications.	5. Total number of arithmetic operations are $N\log_2 N$ complex additions and $(N/2)\log_2 N$ complex multiplications.

Q1.18 What are direct (or slow) convolution and fast convolution?

The response of an LTI system is given by convolution of input and impulse response. The computation of the response of the LTI system by convolution sum formula is called slow convolution because it involves very large number of calculations. The number of calculations in DFT computations can be reduced to a very large extent by FFT algorithms. Hence, computation of the response of the LTI system by FFT algorithm is called fast convolution.

Q1.19 Why is FFT needed?

FFT is needed to compute DFT with reduced number of calculations. The DFT is required for spectrum analysis and filtering operations on the signals using digital computers.

Q1.20 What is bin spacing?

Solution:

The N-point DFT of x(n) is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$$

where, $W_N^{nk} = \left(e^{-j2\pi}\right)^{\frac{nk}{N}}$ is the phase factor or twiddle factor.

The phase factors are equally spaced around the unit circle at frequency increments of F_s/N, where F_s is the sampling frequency of the time domain signal. This frequency increment or resolution is called bin spacing. (The X(k) consists of N-numbers of frequency samples whose discrete frequency locations are given by $f_k = kF_s/N$, for $k = 0, 1, 2, \ldots.., N - 1$).

1.15 MATLAB Programs

Program 1.1

Write a MATLAB program to generate the standard discrete time signals unit impulse, unit step and unit ramp signals.

```
%****************** program to plot some standard signals

n=-20 : 1 : 20;                   %specify the range of n

%****************** unit impulse signal
x1=1;
x2=0;
x=x1.*(n==0)+x2.*(n~=0);          %generate unit impulse signal
subplot(3,1,1);stem(n,x);         %plot the generated unit impulse signal
xlabel('n');ylabel('x(n)');title('unit impulse signal');
%****************** unit step signal
x1=1;
x2=0;
x=x1.*(n>=0)+x2.*(n<0);           %generate unit step signal
subplot(3,1,2);stem(n,x);         %plot the generated unit step signal
xlabel('n');ylabel('x(n)');title('unit step signal');

%****************** unit ramp signal
x1=n;
x2=0;
x=x1.*(n>=0)+x2.*(n<0);           %generate unit ramp signal
subplot(3,1,3);stem(n,x);         %plot the generated unit ramp signal
xlabel('n');ylabel('x(n)');title('unit ramp signal');
```

OUTPUT

The output waveforms of program 1.1 are shown in Fig P1.1.

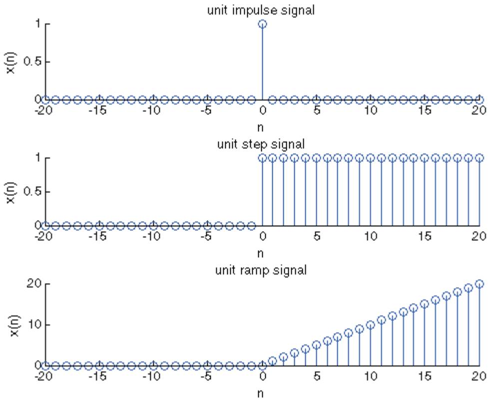


Fig P1.1: *Output waveforms of program 1.1.*

Program 1.2

Write a MATLAB program to generate the standard discrete time signals exponential and sinusoidal signals.

```
%****************** program to plot some standard signals

n=-20 : 1 : 20;                    %specify the range of n

%****************** exponential signal
A=0.95;
x=A.^n;                            %generate exponential signal
subplot(2,1,1);stem(n,x);          %plot the generated exponential signal
xlabel('n');ylabel('x(n)');title('exponential signal');

%****************** sinusoidal signal
N=20;                              %declare periodicity
f=1/20;                            %compute frequency
x=sin(2*pi*f*n);                   %generate sinusoidal signal
subplot(2,1,2);stem(n,x);          %plot the generated sinusoidal signal
xlabel('n');ylabel('x(n)');title('sinusoidal signal');
```

OUTPUT

The output waveforms of program 1.2 are shown in Fig P1.2.

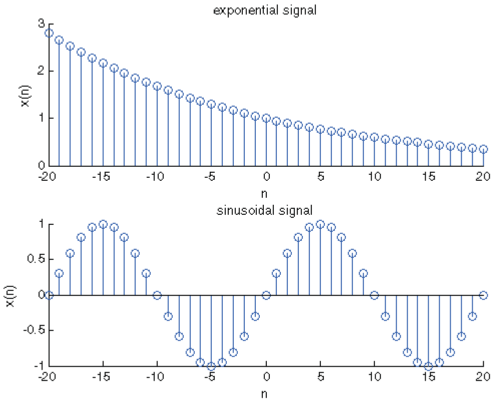


Fig P1.2: *Output waveforms of program 1.2.*

Program 1.3

Write a MATLAB program to perform convolution of the following two discrete time signals.

```
x1(n)=1; 1<n<10        x2(n)=1;   2<n<10
%*****************Program to perform convolution of two signals
%*****************x1(N)=1; n= 1 to 10 and x2(n)=1; n= 2 to 10

n = 0 : 1 : 15;               %specify range of n

x1=1.*(n>=1 & n<=10);         %generate signal x1(n)
x2=1.*(n>=2 & n<=10);         %generate signal x2(n)
N1=length(x1);
N2=length(x2);
x3=conv(x1,x2);               %perform convolution of signals x1(n) and x2(n)
n1=0 : 1 : N1+N2-2;           %specify range of n for x3(n)
subplot(3,1,1);stem(n,x1);
xlabel(‘n’);ylabel(‘x1(n)’);
title(‘signal x1(n)’);

subplot(3,1,2);stem(n,x2);
xlabel(‘n’);ylabel(‘x2(n)’);
title(‘signal x2(n)’);

subplot(3,1,3);stem(n1,x3);
xlabel(‘n’);ylabel(‘x3(n)’);
title(‘signal, x3(n) = x1(n)*x2(n)’);
```

OUTPUT

The input and output waveforms of program 1.3 are shown in Fig P1.3.

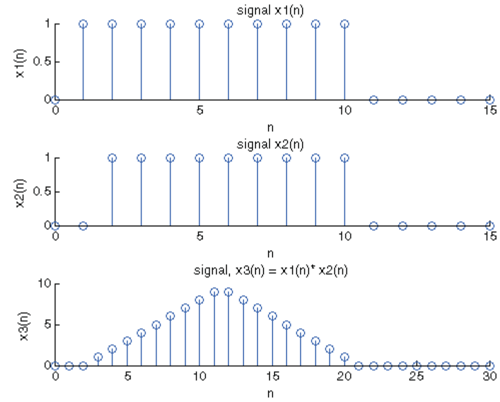


Fig P1.3: *Output waveforms of program 1.3.*

Program 1.4

Write a MATLAB program to perform circular convolution of the discrete time sequences x1(n)={0,1,0,1} and x1(n)={1,2,1,2} using DFT.

```
% Program to perform Circular Convolution via DFT
```

```
clear all
clc

N = 4;               % declare the value of N
x1 = [0,1,0,1];      % declare the input sequences
x2 = [1,2,1,2];

disp('The 4-point DFT of  x1(n) is,');
X1 = fft(x1,N)       % compute 4-point DFT of x1(n)

disp('The 4-point DFT of  x2(n) is,');
X2 = fft(x2,N)       % compute 4-point DFT of x2(n)

disp('The product of DFTs is,');
X1X2 = X1.*X2        % product of DFTs

disp('Circular convolution of x1(n) and x2(n) is,');
X3 = ifft(X1X2)      % perform IDFT to get result of circular convolution
```

OUTPUT

```
The 4-point DFT of  x1(n) is,
X1 =
     2     0    -2     0

The 4-point DFT of  x2(n) is,
X2 =
     6     0    -2     0

The product of DFTs is,
X1X2 =
    12     0     4     0

Circular convolution of x1(n) and x2(n) is,
X3 =
    4      2     4     2
```

Note: Verify the above result with example 1.8.

Program 1.5

Write a MATLAB program to perform 16-point DFT of the discrete time sequence x(n)={1/3,1/3,1/3} and sketch the magnitude and phase spectrum.

```
% program to find DFT and frequency spectrum

clear all
clc

N = 16;                  % specify the length of the DFT
j = sqrt(-1);

xn = zeros (1,N);        % initialize input sequence as zeros
xn(1) = 1/3;             %let given sequence be first three samples
xn(2) = 1/3;
xn(3) = 1/3;
Xk = zeros (1,N);        %initialize output sequence as zeros

for k = 0:1:N-1          % compute DFT
    for n = 0:1:N-1
    Xk(k+1) = Xk(k+1)+xn(n+1)*exp(-j*2*pi*k*n/N);
    end
end

disp ('The DFT sequence is,'); Xk
```

```
disp ('The Magnitude sequence is,');MagXk = abs(Xk)
disp ('The Phase sequence is,');PhaXk = angle(Xk)

Wk=0:1:N-1;              %specify a discrete frequency vector

subplot(2,1,1)
stem(Wk,MagXk);
title('Magnitude spectrum')
xlabel('k'); ylabel('MagXk')

subplot(2,1,2)
stem(Wk,PhaXk);
title('Phase spectrum')
xlabel('k'); ylabel('PhaXk')
```

OUTPUT

```
The DFT sequence is,

Xk =

  Columns 1 through 7

  1.0000          0.8770 - 0.3633i     0.5690 - 0.5690i      0.2252 - 0.5437i
  0 - 0.3333i    -0.0299 - 0.0723i     0.0976 + 0.0976i

  Columns 8 through 14

  0.2611 + 0.1081i    0.3333  + 0.0000i   0.2611 - 0.1081i   0.0976 - 0.0976i
  -0.0299 + 0.0723i  -0.0000  + 0.3333i   0.2252 + 0.5437i

  Columns 15 through 16

  0.5690 + 0.5690i   0.8770 + 0.3633i

The Magnitude sequence is,

MagXk =

  Columns 1 through 12

  1.0000  0.9493   0.8047   0.5885  0.3333   0.0782   0.1381  0.2826
  0.3333  0.2826   0.1381   0.0782

  Columns 13 through 16

  0.3333   0.5885  0.8047   0.9493

The Phase sequence is,

PhaXk =

  Columns 1 through 12

    0        -0.3927   -0.7854   -1.1781   -1.5708  -1.9635  0.7854  0.3927
    0.0000   -0.3927   -0.7854    1.9635

  Columns 13 through 16

    1.5708   1.1781   0.7854    0.3927
```

The magnitude and phase spectrum of program 1.5 are shown in Fig P1.5.

Note: Verify the above result with example 1.6.

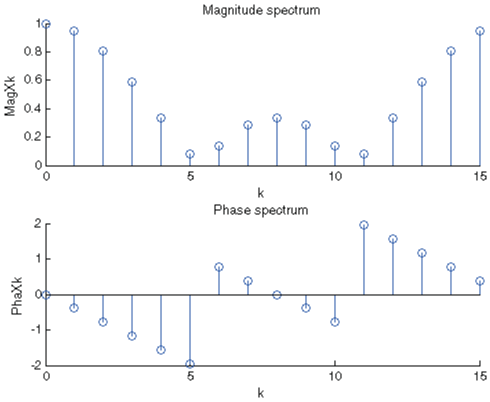


Fig P1.5: *Magnitude and phase spectrum of program 1.5.*

Program 1.6

Write a MATLAB program to perform 8-point DFT of the discrete time sequence x(n)={2,1,2,1,1,2,1,2} and sketch the magnitude and phase spectrum.

```
% program to find DFT and frequency spectrum
clear all
clc
N = 8;                          % specify the length of the DFT
j=sqrt(-1);
xn = [2,1,2,1,1,2,1,2];         % input sequence
Xk = zeros (1,N);               % initialize output sequence as zeros

for k = 0:1:N-1                 % compute DFT
    for n = 0:1:N-1

    Xk(k+1) = Xk(k+1)+xn(n+1)*exp(-j*2*pi*k*n/N);
    end
end

disp (‘The DFT sequence is,’); Xk
disp (‘The Magnitude sequence is,’);MagXk = abs(Xk)
disp (‘The Phase sequence is,’);PhaXk = angle(Xk)

Wk=0:1:N-1;                     % specify a discrete frequency vector
subplot(2,1,1)
stem(Wk,MagXk);
title(‘Magnitude spectrum’)
xlabel(‘k’); ylabel(‘MagXk’)
subplot(2,1,2)
stem(Wk,PhaXk);
title(‘Phase spectrum’)
xlabel(‘k’); ylabel(‘PhaXk’)
```

OUTPUT

```
The DFT sequence is,
Xk =
12.0000       1.0000 + 0.4142i  -0.0000 - 0.0000i     1.0000 + 2.4142i
0 - 0.0000i   1.0000 - 2.4142i  -0.0000 - 0.0000i     1.0000 - 0.4142i
The Magnitude sequence is,
MagXk =
12.0000  1.0824  0.0000   2.6131   0.0000  2.6131  0.0000  1.0824
The Phase sequence is,
PhaXk =
0  0.3927   -2.3201  1.1781  -1.5708  -1.1781 -2.9644  -0.3927
```

The magnitude and phase spectrum of program 1.6 are shown in Fig P1.6.

Note: Verify the above results with example 1.10.

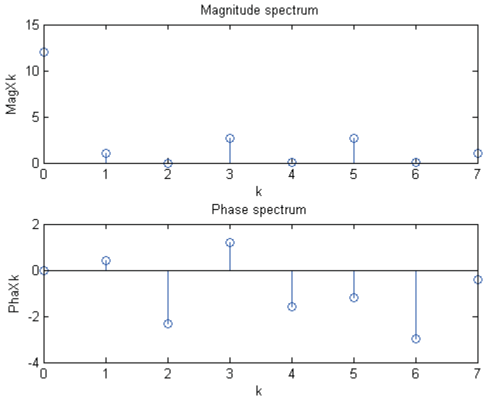


Fig P1.6: *Magnitude and phase spectrum of program 1.6.*

Program 1.7

Write a MATLAB program to perform inverse DFT. Take the frequency domain output sequence of Program 1.6 as input.

```
% program to compute N-point inverse DFT

clear all
clc
N = 8;                     % declare the length of the inverse DFT
j=sqrt(-1);
                           % Xk is input sequence
xk = [12, 1+j*0.4142,0,1+j*2.4142,0,1-j*2.4142,0,1-j*0.4142];
xn = zeros (1,N); %initialize output sequence as zeros

for n= 0:1:N-1             % compute inverse DFT
for k = 0:1:N-1
xn(n+1) = xn(n+1)+(Xk(k+1)*exp(j*2*pi*n*k/N))/N;
end
end
disp('The inverse DFT sequence is,' ); xn
```

OUTPUT

```
The inverse DFT sequence is,
xn =
2.0000 + 0.0000i 1.0000 + 0.0000i 2.0000 - 0.0000i  1.0000 + 0.0000i
1.0000 + 0.0000i 2.0000 – 0.0000i 1.0000 + 0.0000i  2.0000 – 0.0000i
```

Program 1.8

Write a MATLAB program to perform 4-point DFT of the discrete time sequence x(n)={1,1,2,3} using function FFT and sketch the magnitude and phase spectrum.

Also perform inverse DFT on the frequency domain sequence using function IFFT to extract the time domain sequence.

```
% program to demonstrate DFT and inverse DFT Computation using FFT

clear all
clc

N = 4;                          % specify the value of N
xn = [1,1,2,3];                 % input Sequence

disp('DFT of the sequence xn is, ')
Xk = fft(xn,N)                  % compute N-point DFT of input

disp('The magnitude sequence is, ')
MagXk = abs(Xk)                 % compute magnitude spectrum

disp('The phase sequence is, ')
PhaXk = angle(Xk)               % compute phase spectrum

disp('inverse DFT of the sequence Xk is, ')
Xn = ifft(Xk)                   % compute inverse DFT

n = 0:1:N-1;                    % declare a discrete time vector
Wk = 0:1:N-1;                   % declare a discrete frequency vector

subplot(2,2,1)                  % Plot the input sequence
stem(n,xn)
title(' Input sequence')
xlabel('n'); ylabel('xn')

subplot(2,2,2)
stem(n,Xn)
title('inverse DFT sequence')   % Plot the inverse DFT sequence
xlabel('n'); ylabel('Xn')

subplot(2,2,3)                  % Plot the magnitude spectrum
stem(Wk,MagXk)
title('Magnitude spectrum')
xlabel('k'); ylabel('MagXk')

subplot(2,2,4)                  % Plot the frequency spectrum
stem(Wk,PhaXk)
title('Phase spectrum')
xlabel('k'); ylabel('PhaXk')
```

OUTPUT

```
DFT of the sequence xn is,
Xk =
7.0000    -1.0000 + 2.0000i     -1.0000      -1.0000 - 2.0000i
The magnitude sequence is,
MagXk =
7.0000    2.2361     1.0000     2.2361
The phase sequence is,
PhaXk =
0    2.0344    3.1416   -2.0344
inverse DFT of the sequence Xk is,
Xn =
1       1        2        3
```

The input sequence, inverse DFT sequence, magnitude spectrum, and phase spectrum of program 1.8 are shown in Fig P1.8.

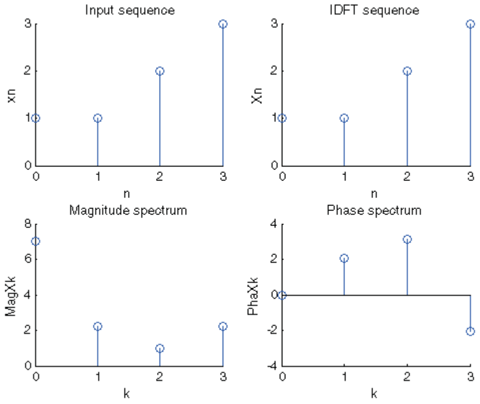


Fig P1.8: Input sequence, magnitude spectrum and phase spectrum of program 1.8.

1.16 Exercises

I. Fill in the blanks with appropriate words.

1. The _______ is called aperiodic convolution.
2. The _______ is called periodic convolution.
3. Appending zeros to a sequence in order to increase its length is called _______.
4. The two methods of sectioned convolutions are _______ and _______method.
5. In _______ method of sectioned convolution, overlapped samples of output sequences are _______.
6. In _______ method, the overlapped samples in one of the output sequences are discarded.
7. In an N-point DFT of a finite duration sequence x(n) of length L, the value of N should be such that _______.
8. The N-point DFT of a L-point sequence will have a periodicity of _______.
9. The convolution property of DFT says that $\mathcal{DFT}\{x(n) \circledast h(n)\}$ = _______.
10. The N-point DFT of a sequence is given by Z-transform of the sequence at N equally spaced points around the _______ in the z-plane.

11. Convolution by FFT is called _______.
12. Convolution using convolution sum formula is called _______.
13. Appending zeros to a sequence in order to increase its length is called _______.
14. In DFT computation using radix-2 FFT, the value of N should be such that _______.
15. The number of complex additions and multiplications in radix-2 FFT are _______ and _______ , respectively.
16. The number of complex additions and multiplications in direct DFT are _______ and _______ , respectively.
17. In 8-point DFT by radix-2 FFT there are _______ stages of computations with _______ butterflies per stage.
18. In _______ butterfly diagram the _______ is multiplied after add-subtract operations.

Answers

1. linear convolution
2. circular convolution
3. zero padding
4. overlap add, overlap save
5. overlap add, added
6. overlap save
7. $N \geq L$
8. N-samples
9. X(k) H(k)
10. unit circle
11. fast convolution
12. slow convolution
13. zero padding
14. $N = 2^m$
15. $N\log_2 N$, $(N/2)\log_2 N$
16. $N(N-1)$, N^2
17. four, four
18. DIF, phase factor

II. State whether the following statements are True or False.

1. Discrete signals are continuous functions of an independent variable.
2. In a digital signal the magnitudes of the signal are unquantized.
3. A discrete time signal x(n) is defined for noninteger values of n.
4. In linear convolution, the length of the input sequences should be the same.
5. In circular convolution, the length of the input sequences need not be the same.
6. The DFT of a sequence is a continuous function of ω.
7. The DFT of a signal can be obtained by sampling one period of Fourier transform of the signal.
8. In sampling $X(e^{j\omega})$, the value of sample at $\omega = 0$ is the same as the value of sample at $\omega = 2\pi$.
9. The DFT of even sequence is purely imaginary and DFT of odd sequence is purely real.
10. In a DFT of real sequence, the real component is even and imaginary component is odd.
11. The multiplication of the DFTs of two sequences is equal to the DFT of the linear convolution of the two sequences.
12. The DFT supports only circular convolution.
13. In FFT algorithm, the N-point DFT is decomposed into successively smaller DFTs.
14. In N-point DFT using radix-2 FFT, the decimation is performed m times, where $m = \log_2 N$.
15. Both DIT and DIF algorithms involve the same number of computations.
16. Bit reversing is required for both DIT and DIF algorithms.

Answers

1. False
2. False
3. False
4. False
5. False
6. False
7. True
8. True
9. False
10. True
11. False
12. True
13. True
14. True
15. True
16. True

III. Choose the right answer for the following questions.

1. ***Sectioned convolution is performed if one of the sequence is very much larger than the other in order to overcome,***

   a) long delay in getting output **b)** larger memory space requirment

   c) both a and b **d)** none of the above

2. ***In overlap save method, the convolution of various sections is performed by,***

   a) zero padding **b)** linear convolution **c)** circular convolution **d)** both b and c

3. ***In N-point DFT of L-point sequence, the value of N to avoid aliasing in frequency spectrum is,***

   a) $N \neq L$ **b)** $N \leq L$ **c)** $N \geq L$ **d)** $N = L$

4. ***The inverse DFT of x(n) can be expressed as,***

   a) $x(n) = \frac{1}{N}\sum_{k=0}^{N} X(k)\, e^{-j\frac{2\pi kn}{N}}$ **b)** $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}$

   c) $x(n) = \frac{1}{N}\sum_{n=0}^{N} X(n)\, e^{-\frac{j2\pi kn}{N}}$ **d)** $x(n) = N\sum_{n=0}^{N-1} X(k)\, e^{-j\frac{2\pi kn}{N}}$

5. ***If DFT {x(n)} = X(k), then DFT*** $\{x(n+m)_N\}$

   a) $X(k)\, e^{-\frac{j2\pi km}{N}}$ **b)** $X(k)\, e^{-\frac{j2\pi k}{mN}}$ **c)** $X(k)\, e^{\frac{j2\pi km}{N}}$ **d)** $X(k)\, e^{\frac{j2\pi k}{mN}}$

6. ***The DFT of product of two discrete time sequences*** $x_1(n)$ ***and*** $x_2(n)$ ***is equivalent to,***

   a) $\frac{1}{N}[X_1(k) \circledast X_2(k)]$ **b)** $\frac{1}{N}[X_1(k)\, X_2(k)]$

   c) $\frac{1}{N}[X_1(k) \circledast X_2^*(k)]$ **d)** $X_1(k) \circledast X_2(k)$

7. ***By correlation property, the DFT of circular correlation of two sequences x(n) and y(n) is,***

   a) $X(k)Y^*(k)$ **b)** $X(k) \circledast Y(k)$ **c)** $X(k) \circledast Y^*(k)$ **d)** $X(k)\, Y(k)$

8. ***The N-point DFT of a finite duration sequence can be obtained as,***

   a) $X(k) = X(z)\Big|_{z = e^{\frac{j2\pi n}{N}}}$ **b)** $X(k) = X(z)\Big|_{z = e^{\frac{j2\pi k}{N}}}$

   c) $X(k) = X(z)\Big|_{z = e^{-\frac{j2\pi kn}{n}}}$ **d)** $X(k) = X(z)\Big|_{z = e^{\frac{j2\pi kn}{N}}}$

9. ***In an N-point sequence, if N = 16, the total number of complex additions and multiplications using Radix-2 FFT are,***

   a) 64 and 80 **b)** 80 and 64 **c)** 64 and 32 **d)** 24 and 12

10. ***The complex valued phase factor/twiddle factor,*** W_N ***can be represented as,***

    a) $e^{-j2\pi N}$ **b)** $e^{-\frac{j2\pi}{N}}$ **c)** $e^{-j2\pi}$ **d)** $e^{-j2\pi KN}$

11. The phase factors are multiplied before the add and subtract operations in,

a) DIT radix-2 FFT
b) DIF radix-2 FFT
c) inverse DFT
d) both a and c

12. If X(k) consists of N-number of frequency samples, then its discrete frequency locations are given by,

a) $f_k = \frac{kF_s}{N}$
b) $f_k = \frac{F_s}{N}$
c) $f_k = \frac{kN}{F_s}$
d) $f_k = N$

Answers

1. c
2. c
3. c
4. b
5. c
6. a
7. a
8. b
9. c
10. b
11. a
12. a

IV. Answer the following questions.

1. Define discrete and digital signal.
2. Define circular convolution.
3. What is the importance of linear and circular convolution in signals and systems?
4. How will you perform linear convolution via circular convolution?
5. What is sectioned convolution? Why is it performed?
6. What are the two methods of sectioned convolution?
7. Define DFT of a discrete time sequence.
8. Define inverse DFT.
9. What is the relation between DTFT and DFT?
10. What is the drawback in Fourier transform and how can it be overcome?
11. List any four properties of DFT.
12. State and prove the shifting property of DFT.
13. What is FFT?
14. What is radix-2 FFT?
15. How many multiplications and additions are involved in radix-2 FFT?
16. What is DIT radix-2 FFT?
17. What is phase factor or twiddle factor?
18. Draw and explain the basic butterfly diagram or flow graph of DIT radix-2 FFT.
19. What are the phase factors involved in the third stage of computation in the 8-point DIT radix-2 FFT?
20. What is DIF radix-2 FFT?
21. Draw and explain the basic butterfly diagram or flow graph of DIF radix-2 FFT.
22. What are the phase factors involved in the first stage of computation in 8-point DIF radix-2 FFT?
23. How will you compute inverse DFT using radix-2 FFT algorithm?
24. What is magnitude and phase spectrum?

V. Solve the following problems.

E1.1 ***Perform circular convolution of the two sequences,***

a) $x_1(n) = \{1,\ 2,\ -1,\ -1\}$; $x_2(n) = \{2,\ 4,\ 6,\ 8\}$

b) $x_1(n) = \{0,\ 0.6,\ -1,\ 1.5,\ 2\}$; $x_2(n) = \{-2,\ 3,\ 0.2,\ 0.7,\ 0.8\}$

E1.2 ***The input x(n) and impulse response h(n) of an LTI system are given by,***

$x(n) = \{-1,\ \underset{\uparrow}{1},\ -1,\ 1,\ -1,\ 1\}$; $h(n) = \{\underset{\uparrow}{-0.5},\ 0.5,\ -1,\ 0.5,\ -1,\ -2\}$

Find the response of the system using

a) Linear convolution,

b) Circular convolution.

E1.3 ***Perform linear convolution of the following sequences by,***

a) Overlap add method

b) Overlap save method

$x(n) = \{1,\ -1,\ 2,\ 1,\ -1,\ 2,\ 1,\ 2,\ 1,\ -1,\ 2\}$; $h(n) = \{2,\ 3,\ -1\}$

E1.4 ***Compute 4-point DFT and 8-point DFT of causal sequence given by,***

$$x(n) = \frac{1}{8} \ ; \ 0 \le n \le 3$$
$$= 0 \ ; \ \text{else}$$

E1.5 ***Compute DFT of the sequence, x(n) = { 0, 2, 3, −1}. Sketch the magnitude and phase spectrum.***

E1.6 ***Compute DFT of the sequence, x(n) = {1, 3, 3, 3 }. Sketch the magnitude and phase spectrum***

E1.7 ***Compute circular convolution of the following sequences using DFT.***

$x_1(n) = \{\underset{\uparrow}{-1},\ 2,\ -2,\ -1\}$ *and* $x_2(n) = \{\underset{\uparrow}{1},\ -2,\ -1,\ -2\}$

E1.8 ***Compute linear and circular convolution of the following sequences using DFT.***

x(n) = {1, 0.2, –1 } and h(n) = {1, –1, 0.2 }.

E1.9 ***Compute 8-point DFT of the discrete time signal, x(n) = {1, 2, 1, 2, 1, 3, 1, 3 },***
a) using radix-2 DIT FFT and b) using radix-2 DIF FFT.

Also sketch the magnitude and phase spectrum.

E1.10 ***In an LTI system the input, x(n) = {1, 2, 1 } and the impulse response, h(n) = {1, 3}. Determine the response of LTI system by radix-2 DIT FFT.***

E1.11 ***Compute the DFT and plot the magnitude and phase spectrum of the discrete time sequence, x(n) = {4, 4, 0, 2 } and verify the result using the inverse DFT.***

E1.12 ***Determine the response of LTI system when the input sequence, x(n) = {–2, –1, –1, 0, 2} by radix 2 DIT FFT. The impulse response of the system is h(n) = {1, –1, –1, 1 }.***

Answers

E1.1 **a)** $x_3(n) = \{\underset{\uparrow}{8},\ 14,\ 4,\ -6\}$; **b)** $x_3(n) = \{\underset{\uparrow}{6.08},\ -0.55,\ 6.4,\ -4.28,\ 0.72\}$

E1.2 $y(n) = \{0.5,\ \underset{\uparrow}{-1},\ 2,\ -2.5,\ 3.5,\ -1.5,\ 1,\ -0.5,\ -0.5,\ 1,\ -2\}$

E1.3 **a)** Overlap add method : $y(n) = \{2,\ 1,\ 0,\ 9,\ -1,\ 0,\ 9,\ -1,\ 0,\ 7,\ -2\}$

b) Overlap save method : $y(n) = \{\times,\ \times,\ 0,\ 9,\ -1,\ 0,\ 9,\ -1,\ 0,\ 7,\ -2\}$

E1.4 4-point DFT ; $X(k) = \{\ 0.5, 0, 0, 0\ \}$

8-point DFT ; $X(k) = \{\ 0.5\angle 0, 0.326\angle -0.374\pi, 0, 0.135\angle -0.125\pi, 0, 0.135\angle 0.125\pi\ , 0, 0.326\angle 0.374\pi\ \}$

E1.5 $X(k) = \{\ 4\ \angle 0,\ 4.243\ \angle -0.75\pi, 2\ \angle 0,\ 4.243\ \angle 0.75\pi\ \}$

$|X(k)| = \{\ 4, 4.243, 2, 4.243\}$

$\angle X(k) = \{\ 0, -0.75\pi, 0, 0.75\pi\ \}$

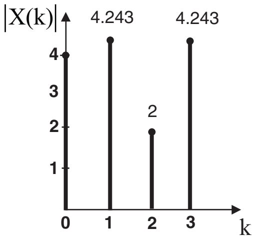


Fig E1.5.1: *Magnitude spectrum.*

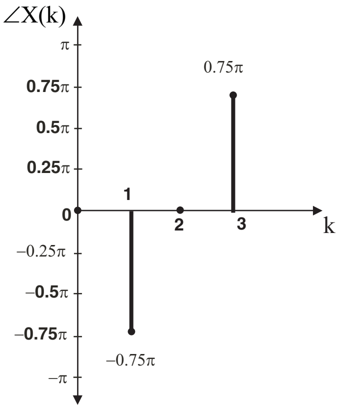


Fig E1.5.2: *Phase spectrum.*

E1.6 $X(k)\ = \{\ 10\angle 0,\ 2\angle -\pi, 2\angle \pi,\ 2\angle \pi\ \}$

$|X(k)| = \{\ 10, 2, 2, 2\}$

$\angle X(k) = \{\ 0, \pi, \pi, \pi\ \}$

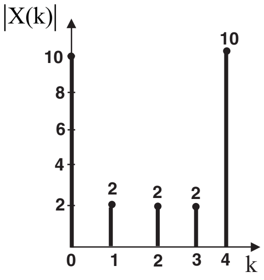


Fig E1.6.1: *Magnitude spectrum.*

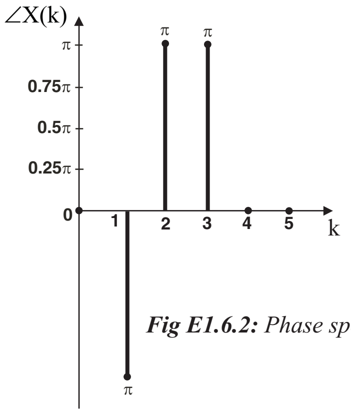


Fig E1.6.2: *Phase spectrum.*

E1.7 $x_1(n) * x_2(n) = \{ -1, 9, -3, 3\}$

E1.8 $x(n) * h(n) = \{ 1, -0.8, -1, 1.04, -0.2 \}$

$x(n) \circledast h(n) = \{2.04, -1, -1\}$

E1.9 $X(k) = \{ 14, j1.414, 0, j1.414, -6, -j1.414, 0, -j1.414 \}$

$= \{ 14, 1.414 \angle 0.5\pi, 0, 1.414 \angle 0.5\pi, 6 \angle \pi, 1.414 \angle -0.5\pi, 0, 1.414 \angle -0.5\pi \}$

$|X(k)| = \{ 14, 1.414, 0, 1.414, 6, 1.414, 0, 1.414 \}$

$\angle X(k) = \{ 0, 0.5\pi, 0, 0.5\pi, \pi, -0.5\pi, 0, -0.5\pi \}$

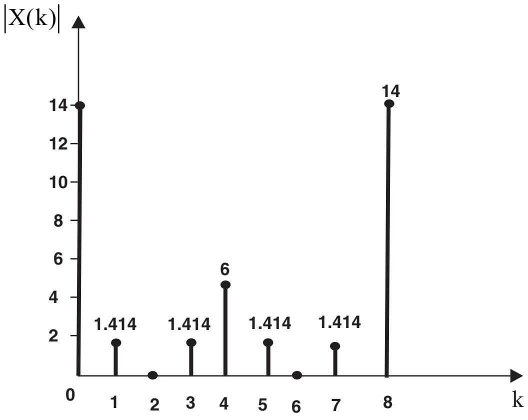


Fig E1.9.1: *Magnitude spectrum.*

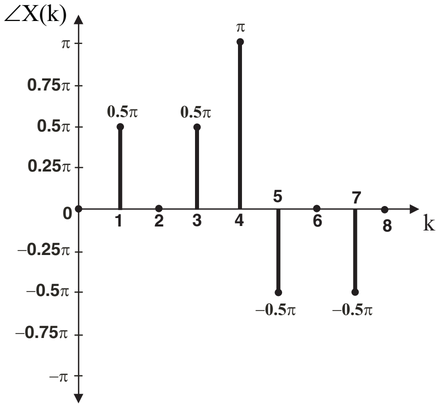


Fig E1.9.2: *Phase spectrum.*

E1.10 $y(n) = \{1, 5, 7, 3 \}$

E1.11 $X(k) = \{ 10, 4 - j2, -2, 4 + j2 \} = \{ 10 \angle 0, 4.472 \angle -0.15\pi, 2 \angle \pi, 4.472 \angle 0.15\pi \}$

$|X(k)| = \{ 10, 4.472, 2, 4.472\}$; $\angle X(k) = \{ 0, -0.15\pi, \pi, 0.15\pi \}$

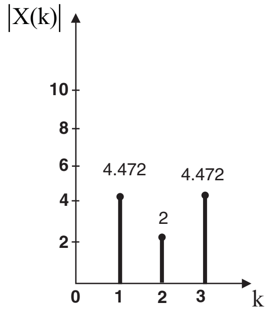


Fig E1.11.1: *Magnitude spectrum.*

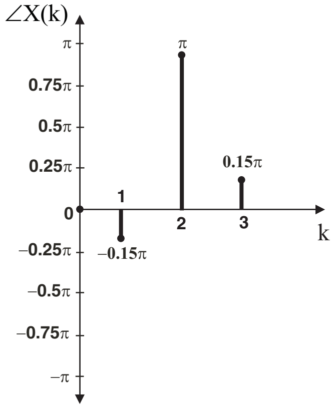


Fig E1.11.2: *Phase spectrum.*

E1.12 $y(n) = \{-2, 1, 2, 0, 2, -3, -2, 2 \}$