Suppose that there are large number of particles in a system whose masses are m_1, m_2, m_3, …. The system can be a *rigid body* in which the particles are in fixed position with respect to each other, or it can be a collection of particles in which there may be all kinds of internal motion.

Let us suppose that the particles of the system are interacting with each other and are also acted by external forces. If $\boldsymbol{p}_1 = m_1\boldsymbol{v}_1$, $\boldsymbol{p}_2 = m_2\boldsymbol{v}_2, \ldots, \boldsymbol{p}_n = m_n\boldsymbol{v}_n$ are the momenta of particles of masses $m_1, m_2, \ldots, m_n$ respectively, then the total momentum ($\boldsymbol{p}$) of the system of the particles is the vector sum of the momentum of individual particles, i.e.

$$\begin{aligned}\boldsymbol{p} &= \boldsymbol{p}_1 + \boldsymbol{p}_2 + \ldots + \boldsymbol{p}_n \\ &= m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 + \ldots + m_n\boldsymbol{v}_n \qquad (1.38)\end{aligned}$$

Differentiating Eq. (1.38) with respect to time t, we have

$$\begin{aligned}\frac{d\boldsymbol{p}}{dt} &= \frac{d\boldsymbol{p}_1}{dt} + \frac{d\boldsymbol{p}_2}{dt} + \ldots + \frac{d\boldsymbol{p}_n}{dt} \\ &= \boldsymbol{F}_1 + \boldsymbol{F}_2 + \ldots + \boldsymbol{F}_n \qquad (1.39)\end{aligned}$$

where $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ are the forces acting on the particles $m_1, m_2, \ldots, m_n$ respectively.

We may note that these forces include external as well as internal forces. However, in accordance with Newton's third law, the internal forces exist in pairs of equal and opposite forces and they balance each other and hence they do not contribute any thing to the external force. This means that right hand side of Eq. (1.39) represents the result and force F_{ext} only due to the *external forces* acting on all the particles of the system. Thus, the sum of external forces is

$$\boldsymbol{F}_{ext} = \frac{d\boldsymbol{p}}{dt} + \frac{d}{dt}(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \ldots + \boldsymbol{p}_n)$$

If the resultant external force is zero, then

$$\frac{d\boldsymbol{p}}{dt} = 0 \text{ or } \boldsymbol{p} = \text{a constant}$$

i.e. $$\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \ldots + \boldsymbol{p}_n = \text{a constant} \qquad (1.40)$$

Thus, if the *resultant external force acting on a system of particles is zero the total linear momentum of the system remains constant*. This simple but quite general result is called the *law of conservation of linear momentum* for a system of particles. We may note that momentum of individual particle may change but their sum, i.e. total momentum remains unaltered in the absence of external forces.

The law of conservation of momentum is a fundamental and exact law of nature and so far no violations of it have even been reported.

Law of Conservation of Angular Momentum

The angular momentum of a particle is defined as the moment of its linear momentum. Mathematically, the angular momentum $\boldsymbol{J}$ of a particle about a point is defined by

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} = m(\boldsymbol{r} \times \boldsymbol{v}) \qquad (1.41)$$

where $\boldsymbol{r}$ is the vector distance of the particle from that point and $\boldsymbol{p} = m\boldsymbol{v}$ is the momentum in an inertial frame in which the point is stationary.

Differentiating Eq. (1.41) with respect to time t, one obtains

$$\frac{d\boldsymbol{J}}{dt} = \frac{d}{dt}(\boldsymbol{r}\times\boldsymbol{p}) = \frac{d\boldsymbol{r}}{dt}\times\boldsymbol{p} + \boldsymbol{r}\times\frac{d\boldsymbol{p}}{dt}$$

or $$\frac{d\boldsymbol{J}}{dt} = \boldsymbol{r}\times\frac{d\boldsymbol{p}}{dt} = \boldsymbol{r}\times\boldsymbol{F}$$

$$\left[\because \frac{d\boldsymbol{r}}{dt}\times\boldsymbol{p} = \boldsymbol{v}\times m\boldsymbol{v} = 0\right]$$

where $\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$ is the force applied on the particle.

The vector product of $\boldsymbol{r}$ and $\boldsymbol{F}$ is called *torque* or *moment of force* about the reference point and is represented by $\boldsymbol{\tau}$. Thus,

$$\tau = \frac{d\boldsymbol{J}}{dt} = \boldsymbol{r}\times\boldsymbol{F} \qquad (1.42)$$

Obviously, the torque is equal to the rate of change of angular momentum ($\boldsymbol{J}$). Its unit is N/m.

Extending the above for a system of n particles in the form of a rigid body, we obtain

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2 + \ldots + \boldsymbol{J}_n$$

ILLUSTRATIVE EXAMPLES

Example 1

A bullet is fired horizontally in the north direction with a velocity of 500 m/s at 30°N latitude. Calculate the horizontal component of Coriolis acceleration and the consequent deflection of the bullet as it hits a target 250 m away. Also find the vertical displacement of the bullet due to gravity.

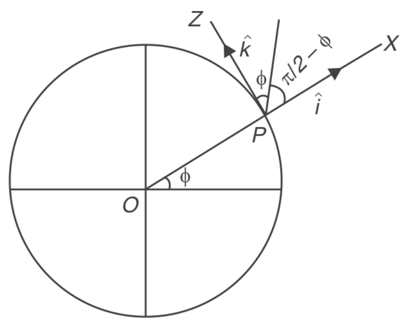

Fig. 1.13

Solution

Let us take X-axis vertically, Z-axis towards north and Y-axis along east. The velocity of the bullet

$$V = 500\,\hat{k}\ \text{m/s}$$

and angular velocity

$$\omega = \omega(\hat{k}\cos 30° + \hat{i}\sin 30°)$$

Since the angular velocity ω of the earth is directed parallel to its axis and is inclined at 30° to the horizontal. Here

$$\omega = \frac{2\pi}{24\times 60\times 60} = 7.2\times 10^{-5}\ \text{rad/s}$$

∵ Coriolis acceleration = $2\omega \times V$

$$= 2\omega(2\hat{k}\cos 30° + \hat{i}\sin 30°)\times 500\hat{k}$$

$$= -2\times 7.2\times 10^{-5}\times 10^{2}\times \frac{1}{2}\hat{j}$$

$$= 0.036\ \text{m/s}^2 \text{ towards west.}$$

Time taken during the journey

$$= 250/500 = 0.5\ \text{s.}$$

∴ Deflection of the bullet due to Coriolis acceleration

$$= -\frac{1}{2}at^2 = \frac{1}{2}\times 0.036\times\left(\frac{1}{2}\right)^2 = 4.5\times 10^{-3}\ \text{m}$$

Now, the vertical displacement of bullet due to gravity

$$= \frac{1}{2}gt^2 = \frac{1}{2}\times 9.8\times\left(\frac{1}{2}\right)^2 = 1.23\,\text{m}$$

Now, Coriolis force = $-2m\omega \times \boldsymbol{V}$

$$= 2\times 0.1\times 7.2\times 10^{-5}\times 5\times 10^{2}\times\frac{1}{2}\hat{i}$$

$$= 3.6\times 10^{-3}\hat{i} = 3.6\times 10^{-3}\ \text{N towards east}$$

Example 2

Prove that the observed acceleration due to gravity g_ϕ at the latitude φ is related to its real value g by the relation

$$g_\phi^2 = \left(g\cos\phi - \omega^2 R\cos\phi\right)^2 + \left(g\sin\phi\right)^2$$

Solution

If the particle is at rest at latitude φ, then it is not acted by Coriolis force.

Thus $\quad \boldsymbol{a}_i = \boldsymbol{a}_r - \omega \times (\omega \times R)$

Now, we take Z-axis along the axis of the earth and X-axis perpendicular to it, then

$$\boldsymbol{a}_r = \boldsymbol{a}_i - \omega\,(\omega \times R)$$

or $$g_\phi = -g\left(\hat{i}\cos\phi + \hat{k}\sin\phi\right) - \omega\hat{k}$$

$$\times\left[\omega\hat{k}\times R\left(\hat{i}\cos\phi + \hat{k}\sin\phi\right)\right]$$

$$= -g\left(\hat{i}\cos\phi + \hat{k}\sin\phi\right) + \omega^2 R\cos\phi\,\hat{i}$$

$$= -\hat{i}\left(g\cos\phi + \omega^2 R\cos\phi\right) - \hat{k}g\sin\phi$$

$$\therefore\ g_\phi^2 = g_\phi\cdot g_\phi\left(g\cos\phi - \omega^2 R\cos\phi\right)^2 + \left(g\sin\phi\right)^2$$

Example 3

A particle of mass 200 gm is stationary in an inertial reference system. Describe and interprete its motion in a frame rotating with angular speed 5π rad/s. The axis of rotation is observers. The Lorentz contraction may be observed, but not 'seen' (since the eye and the instantaneous cameras record pictures formed photons that arrive together).

Time Dilation

Consider a clock at a point x' in an inertial frame S'. Let S' be moving with a velocity v with respect to another inertial frame S along the common X-axis. Let a clock be situated in frame S at x. Let clock in S frame give out signals at two instants of time t_1 and t_2 as measured by an observer in S. Let an observer in frame S' measure these instants of time t_1' and t_2' with his own clock. Then from Lorentz transformation equations, we obtain

$$\left.\begin{aligned} t_1' &= \frac{t_1 - \dfrac{vx}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}} \\ t_2' &= \frac{t_2 - \dfrac{vx}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}} \end{aligned}\right\} \quad (2.24)$$

Subtracting one from the other, we obtain

$$t_2' - t_1' = \frac{t_2 - t_1}{\sqrt{1-\dfrac{v^2}{c^2}}} \quad (2.25)$$

Letting $t_2' - t_1' = \tau$ and $t_2 - t_1 = \tau_0$, we have

$$\tau = \frac{\tau_0}{\sqrt{1-\dfrac{v^2}{c^2}}} \quad (2.26)$$

or

$$\tau_0 = \tau\sqrt{1-\frac{v^2}{c^2}} \quad (2.27)$$

For the observer in frame S, the time interval between the two signals is $\tau_0 = t_2 - t_1$ and for the observer in S' the time interval between the same two signals is $\tau = t_2' - t_1'$. Equation (2.26) reveals that τ is larger than τ_0. Obviously, for the moving observer, the time interval appears to be *elongated* or *dilated*. This phenomenon is called *time dilation.*

2.10 PROPER FRAME, PROPER LENGTH AND PROPER TIME

The inertial frame of reference in which the observed body is at rest is called the proper frame of reference. The length of a rod as measured in the inertial frame in which it is at rest is called the proper length. The relation between the *proper length* (L_0) and the apparent or non-proper length (L) is as follows:

$$L = L_0\sqrt{1-\frac{v^2}{c^2}} \quad (2.28)$$

The time interval recorded by a clock fixed with respect to the observed event is called the proper time interval. The relation between the proper time (τ_0) and apparent or non-proper time (τ) is as follows:

$$\tau = \frac{\tau_0}{\sqrt{1-\dfrac{v^2}{c^2}}} \quad (2.29)$$

Obviously, proper time noted by a moving observer is always less than the corresponding interval of time in a stationary frame. A stationary observer finds that a moving clock runs slower than a stationary one. An event which repeats itself with a certain period in S' will appear to have a longer period when observed from S. It is important to note that this effect is mutually reciprocal between two observers.

The relation (2.29) suggests that any physical process occurring in S' appears, when viewed from S, to have slowed down compared with an identical process occurring in S. Thus, spectrum of radiation emitted by atoms moving relative to a spectroscope will appear shifted towards the longer wavelength or the red side of the corresponding spectrum emitted by atoms which are at rest relative to the spectroscope. This effect due to time dilation has to be considered while considering *Doppler shift in radiation.*

and $$U_z = \frac{U'_z\sqrt{1-\frac{v^2}{c^2}}}{\sqrt{1+\frac{vU'_x}{c^2}}} \quad (2.36)$$

The relativistic velocity transformation equations take the form as $c \to \infty$, i.e.

$$U'_x = U_x,\ U'_y = U_y \text{ and } U'_z = U_z$$

Example 12

A spaceship moving away from the earth with velocity $0.5\,c$ fires a rocket whose velocity relative to spaceship is $0.8\,c$ (a) away from the earth (b) towards the earth. What will be the velocity of the rocket as observed from the earth in the two cases?

Solution

Let us regard the earth as S frame and spaceship as S' frame. The rocket is the object whose velocity in the S frame is to be determined.

We have $$U = \frac{U'+v}{1+\frac{U'v}{c^2}}$$

$$U' = 0.8\,c$$
$$v = 0.5\,c$$

$$\therefore \quad U = \frac{0.8\,c + 0.5\,c}{1+0.8\times 0.5}$$

In the second case, $U' = -0.8\,c$

$$\therefore \quad U = \frac{U'+v}{1+\frac{U'v}{c^2}} = \frac{-0.8\,c+0.5\,c}{1+\{(-0.8)\times(0.5)\}}$$

$$\frac{-0.3\,c}{1-0.4} = -0.5\,c$$

Example 13

In the laboratory frame two particles are observed to travel in opposite directions with speed 2.8×10^8 m/s. Deduce the relative speed of the particles.

Solution

We have $$U'_x = \frac{U_x - v}{1-\frac{U'_x v}{c^2}}$$

$$U_x = 2.8 \times 10^8 \text{ m/s}$$
$$v = -2.8 \times 10^8 \text{ m/s}$$
$$c = 3 \times 10^8 \text{ m/s}$$

$$U'_x = \frac{(2.8\times10^8)-(-2.8\times10^8)}{1-\frac{(2.8\times10^8)(-2.8\times10^8)}{(3\times10^8)^2}}$$

$$= 2.99 \times 10^8 \text{ m/s}$$

Obviously, the velocity of the first particle relative to the second is 2.99×10^8 m/s.

Example 14

A radioactive nucleus while moving with a velocity $0.2\,c$ in the lab-frame emits a β-particle. The β-particle moves with a speed $0.6\,c$ relative to the nucleus. What is the velocity and direction of the β-particle if it is emitted in a direction (a) parallel (b) perpendicular to the direction of motion of the nucleus in the lab-frame?

Solution

Let us consider that the frame of reference fixed on nucleus be S' and the lab-frame be S. The direction of motion of the nucleus be the X-direction.

(a) $$U_x = \frac{U'_x + v}{1+\frac{U'v}{c^2}}$$

$$U'_x = 0.6c$$
$$v = 0.2c$$
$$U'_x = U'_z = 0$$

$$\therefore \quad U_x = \frac{0.6c+0.2c}{1+0.12}$$

$$= \frac{0.8c}{1.12} = 0.174c$$

β-particle is emitted in X-direction

$$U_y = \frac{U'_y\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{U'_x v}{c^2}} = 0$$

Similarly, $U_z = 0.$