Hence, the general solution of the complete equation is

$$y_x = C_1 3^x + C_2 2^x + 1$$

When $y_0 = 1$ and $y_1 = -1$, we get

$$y_0 = C_1 + C_2 + 1 = 1$$

$$y_1 = 3C_1 + 2C_2 + 1 = -1$$

We get $C_1 = -2$ and $C_2 = 2$

Hence, the solution is

$$y_x = -2.3^x + 2^{x+1} + 1.$$

Alternative Method: To find particular solution of the complete equation, we will use hit and trial method. Let $y_x^* = A$ be the solution of the given equation, we get

$$A - 5A + 6A = 2$$

$$A = 1$$

$$y_x^* = A = 1$$

Example 6. *Solve $\Delta y_x + \Delta^2 y_x = \sin x$.*

Solution. The given equation in symbolic form is

$$\{(E-1) + (E-1)^2\}y_x = \sin x$$

$$(E^2 - E)y_x = \sin x$$

$$(E-1)y_{x+1} = \sin x$$

The auxiliary equation is

$$m - 1 = 0, m = 1$$

Then, general solution is $y_x = C_1 1^x = C_1$

Particular solution of $\sin x = \frac{1}{E-1}\sin x$

$$= \text{imaginary part of } \frac{1}{E-1}e^{ix}$$

$$= \text{imaginary part of } \frac{1}{E-1}(e^i)^x$$

$$= \text{imaginary part of } \frac{e^{ix}}{e^i - 1}$$

$$= \text{imaginary part of } \frac{e^{ix}(e^{-i}-1)}{(e^i-1)(e^{-i}-1)}$$

$$= \text{imaginary part of } \frac{e^{i(x-1)} - e^{ix}}{1-(e^i+e^{-i})+1}$$

$$= \frac{\sin(x-1) - \sin x}{2(1-\cos 1)}$$

Hence, the general solution is

$$y_{x+1} = C_1 + \frac{\sin(x-1) - \sin x}{2(1-\cos 1)}$$

$$\Rightarrow \quad y_x = C_1 + \frac{\sin(x-2) - \sin(x-1)}{2(1\sin 1)}.$$

Example 7. *Solve $y_{x+2} + y_x = \sin x\frac{\pi}{2}$.* ...(1)

Solution. The auxiliary equation of the given equation is

$$m^2 + 1 = 0$$

$$\Rightarrow \quad m = \pm 1$$

$$\Rightarrow \quad m = \cos\frac{\pi}{2} \pm i\sin\frac{\pi}{2}$$

The general solution of the reduced equation is

$$y_x = C_1\cos\left(\frac{x\pi}{2} + C_2\right)$$

For particular solution, we will use hit and trial method. Let

$$y_x^* = A\sin\frac{x\pi}{2} + B\sin\frac{x\pi}{2}$$

Substituting this value in equation (1), we get

$$A\sin(x+2)\frac{\pi}{2} + B\sin(x+2)\frac{\pi}{2}$$

$$+ A\sin\frac{\pi x}{2} + B\sin\frac{\pi x}{2} = \sin\frac{\pi x}{2}$$

$$-A\sin\frac{\pi x}{2} - B\sin\frac{\pi x}{2} + A\sin\frac{\pi x}{2}$$

$$+ B\sin\frac{\pi x}{2} = \sin\frac{\pi x}{2}$$

$$0 = \sin\frac{\pi x}{2}$$

$y_x^* = A\sin\frac{x\pi}{2} + B\cos\frac{x\pi}{2}$ fails.

Now, consider

$y_x^* = Ax\sin\frac{x\pi}{2} + Bx\cos\frac{x\pi}{2}$ substituting this in equation (1), we get

$$A(x+2)\sin(x+2)\frac{\pi}{2} + B(x+2)\cos(x+2)\frac{\pi}{2}$$

$$- Ax\sin\frac{x\pi}{2} + Bx\cos\frac{x\pi}{2} = \sin\frac{x\pi}{2}$$

$$- A(x+2)\sin\frac{x\pi}{2} - B(x+2)\cos\frac{x\pi}{2} + Ax\sin\frac{x\pi}{2}$$

$$+ Bx\sin\frac{x\pi}{2} = \sin\frac{x\pi}{2}$$

$$\sin\frac{x\pi}{2}(-2A) + \cos\frac{x\pi}{2}(-2B) = \sin\frac{x\pi}{2}$$

On comparing the coefficient of

$\cos\frac{x\pi}{2}$ and $\sin\frac{x\pi}{2}$, we get

$$-2A = 1, -2B = 0$$

$$A = -\frac{1}{2}, B = 0$$

Hence, the general solution of the complete equation is

$$y_x = 4\cos\left(\frac{x\pi}{2} + C_2\right) - \frac{1}{2}x\sin\frac{x\pi}{2}.$$

Example 8. *Solve $y_{x+2} - 7y_{x+1} - 8y_x = (x^2 - x)2^x$.*

Solution. The auxiliary equation is

$$m^2 - 7m - 8 = 0$$

$$\Rightarrow \quad m = 8, -1$$

$y_2 = y_1 + hf(x_1, y_1) = 1.10000 + 0.1[(1.10000)^2 - (0.1)^2] = 1.22000$

$y_3 = y_2 + hf(x_2, y_2) = 1.22000 + 0.1[(1.22)^2 - (0.2)^2] = 1.22000 - 0.14484 = 1.36484$

$y_4 = y_3 + hf(x_3, y_3) = 1.36484 + 0.1[(1.36484)^2 - (0.3)^2] = 1.54212$

$y_5 = y_4 + hf(x_4, y_4) = 1.54212 + 0.1[(1.54212)^2 - (0.4)^2] = 1.76393$

Hence, the value of y at $x = 0.5$ is 1.76393.

Example 3. *Find the solution of differential equation* $\frac{dy}{dx} = xy$ *with* $y(1) = 5$ *in the interval* $[1, 1.5]$ *using* $h = 0.1$.

Solution. As per given, we have $x_1 = 1, y_0 = 5, f(x, y) = xy$

Using Euler's method

$$y_{n+1} = y_n + hf(x_n, y_n)$$

By considering $n = 0, 1, 2, \ldots$ in succession, we get

For $n = 0$

$y_1 = y_0 + hf(x_0, y_0) = 5 + 0.1[1 \times 5] = 5.5$

For $n = 1$

$y_2 = y_1 + hf(x_1, y_1) = 5.5 + 0.1[1.1 \times 5.5] = 6.105$

For $n = 2$

$y_3 = y_2 + hf(x_2, y_2) = 6.105 + 0.1[1.2 \times 6.105] = 6.838$

For $n = 3$

$y_4 = y_3 + hf(x_3, y_3) = 6.838 + 0.1[1.3 \times 6.838] = 7.727$

For $n = 4$

$y_5 = y_4 + hf(x_4, y_4) = 7.727 + 0.1[1.4 \times 7.727] = 8.809$

Hence, the value of is 8.809.

Example 4. *Find the solution of* $\frac{dy}{dx} = x^2 + y^2, y(0) = 0$ *in the range* $0 \le x \le 0.5$, *using Euler's method.*

Solution. We have $x_0 = 0, y_0 = 0, h = 0.1$ and $0 \le x \le 0.5$

$\therefore \quad x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, x_4 = 0.4, x_5 = 0.5$

Also $f(x, y) = x^2 + y^2$

$\Rightarrow \quad f(x_0, y_0) = f(0, 0) = 0$

$\therefore \quad y_1 = y_0 + hf(x_0, y_0) = 0 + 0.1 \times 0 = 0$

Now $f(x_1, y_1) = f(0.1, 0) = (0.1)^2 + 0^2 = 0.01$

$y_2 = y_1 + hf(x_1, y_1) = 0 + 0.1(0.01) = 0.001$

$f(x_2, y_2) = f(0.2, 0.001) = (0.2)^2 + (0.001)^2 = 0.040001$

$\therefore \quad y_3 = y_2 + hf(x_2, y_2) = 0.001 + 0.1(0.040001) = 0.005$

Now $f(x_3, y_3) = f(0.3, 0.005) = (0.3)^2 + (0.005)^2 = 0.090025$

$\Rightarrow \quad y_4 = y_3 + hf(x_3, y_3) = 0.005 + 0.1(0.090025) = 0.014$

Now $f(x_4, y_4) = f(0.4, 0.014) = (0.4)^2 + (0.014)^2 = 0.160196$

$\Rightarrow \quad y_5 = y_4 + hf(x_4, y_4) = 0.014 + 0.1(0.160196) = 0.03$

Further

$f(x_5, y_5) = f(0.5, 0.03) = (0.5)^2 + (0.03)^2 = 0.2509$

$\Rightarrow \quad y_6 = y_5 + hf(x_5, y_5) = 0.03 + 0.1(0.2509) = 0.055$

Here $y(0.1) = 0.001, y(0.2) = 0.005,$
$y(0.03) = 0.055, y(0.4) = 0.03,$
$y(0.5) = 0.055.$

Example 5. *Apply Euler's method to initial value problem* $\frac{dy}{dx} = x + y, y(0) = 0$, *when* $x = 0$ *to* $x = 1.0$ *taking* $h = 0.2$. (Mumbai-2005, Rohtak-2003)

Solution. We have $h = 0.2, x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8, x_5 = 1.0.$

Also, $f(x, y) = x + y$

By Euler's method,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

and, $f(x_0, y_0) = f(0, 0) = 0 + 0 = 0$

$y_1 = y_0 + hf(x_0, y_0) = 0 + 0.2(0) = 0$

Now, $f(x_1, y_1) = f(0.2, 0) = 0.2 + 0 = 0.2$

$y_2 = y_1 + hf(x_1, y_1) = 0 + 0.2(0.2) = 0.04$

Now, $f(x_2, y_2) = f(0.4, 0.04) = 0.4 + 0.04 = 0.44$

$y_3 = y_2 + hf(x_2, y_2) = 0.04 + 0.2(0.44) = 0.128$

Now,

$f(x_3, y_3) = f(0.6, 0.128) = 0.6 + 0.128 = 0.728$

$y_4 = y_3 + hf(x_3, y_3) = 0.128 + 0.2(0.728) = 0.2736$

Similarly $\quad y^{iv} = F_3 + f_y F_2 + 3F_1(f_{xy} + ff_{yy}) + f_y^2 F_1$

Putting these values in (7), we get

$$y(x+h) = y_x + hf + \frac{h^2}{2}F_1 + \frac{1}{6}h^3(F_2 + f_y F_1) + \frac{1}{24}h^4[F_3 + f_y F_2 + 3(f_{xy} + ff_{yy})F_1 + f_y^2 F_1] + \ldots \quad \ldots(10)$$

Using the above notation and the Taylor's theorems, we get,

$$k_1 = hf \qquad [\text{where } f = f(x,y)]$$

$$k_2 = h\left[f + mhF_1 + \frac{1}{2}m^2h^2F_2 + \frac{1}{6}m^3h^3F_3 + \ldots\right]$$

$$k_3 = h\left[f + mhF_1 + \frac{1}{2}h^2(n^2F_2 + 2mn + f_yF_1) + \frac{1}{6}h^3\{n^3F_3 + 3m^2nf_yF_2 + 6mn^2(f_{xy} + ff_{yy})F_1\} = \ldots\right]$$

and
$$k_4 = h\left[f + phF_1 + \frac{1}{2}h^2(p^2F_2 + 2npf_yF_1) + \frac{1}{6}h^3\{p^3F_3 + 3n^2pf_yF_2 + 6np^2(f_{xy} + ff_{yy})F_1 + 6nmpf_y^2F_1\} + \ldots\right]$$

Substituting these values in (8), we get

$$y(x+h) = y_{(x)} + (a+b+c+d)hf + (bm + cn + dp)h^2F_1 + \frac{1}{2}(bm^2 + cn^2 + dp^2).$$

$$h^2 f_y F_1 + \frac{1}{2}(cm^2n + dn^2p)h^4 f_y F_2 + (cmn^2 + dnp^2)h^4(f_{xy} + ff_{yy})F_1 + dmnph^4 f_y^2 F_1 + 0(h^5) \quad \ldots(11)$$

Equating this with (10), we get

$$a + b + c + d = 1, \qquad cmn + dnp = \frac{1}{6}$$

$$bm + cn + dp = \frac{1}{2}, \qquad cmn^2 + dnp^2 = \frac{1}{8}$$

$$bm^2 + cn^2 + dp^2 = \frac{1}{3}, \qquad cm^2n + dn^2p = \frac{1}{12}$$

$$bm^3 + cn^3 + dp^3 = \frac{1}{4}, \qquad dmnp = \frac{1}{24}.$$

These are eight equations in seven unknowns; A classical solution to these eight equations is

$$m = n = \frac{1}{2}, p = 1, a = d = \frac{1}{6}, b = c = \frac{1}{3}$$

Putting these values in (7) and (8), the Runge-Kutta formulae reduces to

$$\left.\begin{aligned} k_1 &= hf(x,y) \\ k_2 &= hf\left(x + \frac{h}{2}, y + \frac{1}{2}k_1\right) \\ k_3 &= hf\left(x + \frac{h}{2}, y + \frac{1}{2}k_2\right) \\ k_4 &= hf(x+h, y+k_3) \end{aligned}\right\} \quad \ldots(12)$$

and
$$y(x+h) = y_{(x)} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

From this formula, we have

$$y_1 = y(x_0 + h) = y_0 + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4]$$

where
$$\left.\begin{aligned} k_1 &= hf(x_0, y_0) \\ k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\ k_4 &= hf(x_0 + h, y_0 + k_3 \end{aligned}\right\} \quad \ldots(13)$$

REMARK

- $0(h^2)$ means 'terms containing second and higher power of h and is read as order of h^2.'