## Solved Examples

**Example 1.** *Find the $n^{th}$ derivative of $x^2 \sin x$.* (UPTU–2009)

**Solution.** Let $u = \sin x$ and $v = x^2$.

Then,
$$u_n = \sin\left[x + \frac{n\pi}{2}\right]$$
$$u_{n-1} = \sin\left(x + (n-1)\frac{\pi}{2}\right)$$
$$u_{n-2} = \sin\left[x + (n-2)\frac{\pi}{2}\right]$$

Also, $v_1 = 2x,\ v_2 = 2,\ v_3 = 0$

Now, by Leibnitz's theorem, we have
$$\frac{d^n}{dx^n}(uv) = u_n.v + {}^nC_1 u_{n-1}.v_1 + {}^nC_2 u_{n-2}.v_2$$
$$\Rightarrow \frac{d^n}{dx^n}(x^2 \sin x) = \sin\left(x + \frac{n\pi}{2}\right)x^2 + {}^nC_1 \sin\left[x + (n-1)\frac{\pi}{2}\right]2x + {}^nC_2 \sin\left[x + (n-2)\frac{\pi}{2}\right]2$$
$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) + 2nx \sin\left[x + (n-1)\frac{\pi}{2}\right] + n(n-1)\sin\left[x + (n-2)\frac{\pi}{2}\right]$$

**Example 2.** *Find the $n^{th}$ derivative of $x^3 \cos x$.*

**Solution.** Let $u = \cos x$ and $v = x^3$.

Then,
$$u_n = \cos\left[x + \frac{n\pi}{2}\right], \qquad v_1 = 3x^2$$
$$u_{n-1} = \cos\left(x + (n-1)\frac{\pi}{2}\right), \qquad v_2 = 6x$$
$$u_{n-2} = \cos\left[x + (n-2)\frac{\pi}{2}\right], \qquad v_3 = 6$$
$$u_{n-3} = \cos\left[x + (n-3)\frac{\pi}{2}\right], \qquad v_4 = 0$$

Now, by Leibnitz's theorem, we have
$$\frac{d^n}{dx^n}(uv) = u_n.v + {}^nC_1 u_{n-1}.v_1 + {}^nC_2 u_{n-2}.v_2 + {}^nC_3 u_{n-3}.v_3$$
$$\Rightarrow \frac{d^n}{dx^n}(x^3 \cos x) = \cos\left(x + \frac{n\pi}{2}\right)x^3 + {}^nC_1 \cos\left[x + (n-1)\frac{\pi}{2}\right]3x^2 + {}^nC_2 \cos\left(x + (n-2)\frac{\pi}{2}\right)6x + {}^nC_3 \cos\left(x + (n-3)\frac{\pi}{2}\right)6$$
$$= x^3 \cos\left(x + \frac{n\pi}{2}\right) - 3x^2.n\sin\left(x + \frac{n\pi}{2}\right) - 3n(n-1)x\cos\left(x + \frac{n\pi}{2}\right) - n(n-1)(n-2)\sin\left(x + \frac{n\pi}{2}\right)$$
$$= [x^3 - 3n(n-1)x]\cos\left(x + \frac{n\pi}{2}\right) + [3x^2 n - n(n-1)(n-2)]\sin\left(x + \frac{n\pi}{2}\right).$$

**Example 3.** *If $y = a \cos(\log x) + b \sin(\log x)$, show that*
$$x^2 y_2 + xy_1 + y = 0$$
*and* $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$

(UPTU–2004, 2012, Madras–2000)

**Solution.** We have
$$y = a \cos(\log x) + b \sin(\log x) \qquad \ldots(1)$$
Differentiating (1) with respect to $x$, we have
$$y_1 = -\frac{a}{x}\sin(\log x) + \frac{b}{x}\cos(\log x)$$
$$xy_1 = -a\sin(\log x) + b\cos(\log x)$$
Again, differentiating w.r.t. $x$, we get
$$xy_2 + y_1 = -\frac{a}{x}\cos(\log x) - \frac{b}{x}\sin(\log x)$$
$$\Rightarrow \quad x^2 y_2 + xy_1 = -a\cos(\log x) - b\sin(\log x) = -y$$
$$\Rightarrow \quad x^2 y_2 + xy_1 + y = 0 \qquad \ldots(2)$$
Now, differentiating (2) both sides $n$ times by Leibnitz's theorem, we get
$$D^n(x^2 y_2) + D^n(xy_1) + D^n(y) = 0$$
$$\Rightarrow \quad (D^n y_2)x^2 + {}^nC_1(D^{n-1}y_2)(Dx^2) + {}^nC_2(D^{n-2}y_2)(D^2x^2) + (D^n y_1)x + {}^nC_1(D^{n-1}y_1)(Dx) + D^n y = 0$$
$$\Rightarrow x^2 y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2}2y_n + xy_{n+1} + ny_n + y_n = 0$$
$$\Rightarrow \quad x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

**Example 4.** *If $y = e^{a\sin^{-1}x}$, show that*
$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$$

(MDU–1998, VTU–2003, KU–1999)

**Solution.** We have
$$y = e^{a\sin^{-1}x} \Rightarrow y_1 = e^{a\sin^{-1}x}.\frac{a}{\sqrt{1-x^2}}$$
$$y_1\sqrt{1-x^2} = ae^{a\sin^{-1}x} = ay$$
$$\Rightarrow y_1^2(1-x^2) = a^2y^2 \qquad \ldots(1)$$

$$= -\frac{1}{12} \text{ at } \left(\frac{1}{2}, \frac{1}{3}\right).$$

Now, $rt - s^2$ = positive.

Also, $r$ is negative, hence the function $u$ has a maximum at $x = \frac{1}{2}$, $y = \frac{1}{3}$.

The maximum value is

$$= \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}.$$

**Example 3.** *Discuss the maximum or minimum values of u, where* $u = 2a^2xy - 3ax^2y - ay^3 + x^3y + xy^3$.

**Solution.** We have

$$u = 2a^2xy - 3ax^2y - ay^3 + x^3y + xy^3$$

which gives

$$\frac{\partial u}{\partial x} = 2a^2y - 6axy + 3x^2y + y^3$$

and $\dfrac{\partial u}{\partial y} = 2a^2x - 3ax^2 - 3ay^2 + x^3 + 3xy^2$

For a maximum and minima of $u$, we have

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0$$

which gives,

$$y(2a^2 - 6ax + 3x^2 + y^2) = 0 \quad ...(1)$$

and $2a^2x - 3ax^2 - 3ay^2 + x^3 + 3xy^2 = 0$ ...(2)

Equation (1) and (2) gives the following values of $x$ and $y$ :

$$x = 0, y = 0; x = a, y = 0;$$

$$x = 2a, y = 0; x = \frac{3}{2}a, y = \pm\frac{1}{2}a;$$

$$x = a, y = a, x = \frac{1}{2}, y = \frac{1}{2}a;$$

$$x = a, y = -a; x = \frac{1}{2}a, y = -\frac{1}{2}a.$$

Then, we get the following pairs of values of $x$ and $y$ which make the function $u$ stationary.

$$(0,0), (a,0), (2a,0), \left(\frac{3}{2}a, \frac{1}{2}a\right), \left(\frac{3}{2}a, -\frac{1}{2}a\right)$$

$$(a,a), \left(\frac{1}{2}a, \frac{1}{2}a\right), (a,-a), \left(\frac{1}{2}a, -\frac{1}{2}a\right).$$

Also $r = \dfrac{\partial^2 u}{\partial x^2} = -6ay + 6xy,$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 2a^2 - 6ax + 3x^2 + 3y^2,$$

and $t = \dfrac{\partial^2 u}{\partial y^2} = -6ay + 6xy.$

For (0, 0).

$$r = 0, s = 2a^2, t = 0$$

$\Rightarrow$ $rt - s^2$, is negative.

Therefore, we have neither maximum nor a minimum of $u$ at (0, 0).

Similarly, we can easily shown that $u$ has neither a maximum nor a minimum at $(a, 0)$, $(2a, 0)$, $(a, a)$, $(a, -a)$.

For $\left(\dfrac{3a}{2}, \dfrac{a}{2}\right)$.

$$r = \frac{3}{2}a^2, s = \frac{1}{2}a^2, t = \frac{3}{2}a^2,$$

$\Rightarrow$ $rt - s^2$ is positive.

Here, since $r$ is positive, therefore $u$ has minimum at $\left(\dfrac{3a}{2}, \dfrac{a}{2}\right)$.

Similarly, we can check the maxima and minima at all other points.

**Example 4.** *Find the maximum and minimum values of* $xy(a - x - y)$.

**Solution.** Let $u = xy(a - x - y)$

Then $\dfrac{\partial u}{\partial x} = ay - 2xy - y^2$

and $\dfrac{\partial u}{\partial y} = ax - x^2 - 2xy.$

For a maximum or minimum of $u$, we have

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0.$$

Thus, we have

$$ay - 2xy - y^2 = 0 \Rightarrow y(a - 2x - y) = 0 \quad ...(1)$$

$$ax - x^2 - 2xy = 0 \Rightarrow x(a - x - 2y) = 0 \quad ...(2)$$

Solving (1) and (2), we get the following pairs of values $x$ and $y$ which makes the function stationary

$$(0,0), (0,a), (a,0), \left(\frac{1}{3}a, \frac{1}{3}a\right).$$

Here $r = \dfrac{\partial^2 u}{\partial x^2} = -2y$, $s = \dfrac{\partial^2 u}{\partial x \partial y} = a - 2x - 2y$,

and $t = \dfrac{\partial^2 u}{\partial y^2} = -2x.$

**For (0, 0).** $r = 0, s = a, t = 0$

$\Rightarrow$ $rt - s^2$ is negative.

$\therefore$ We have neither a maximum nor a minimum of $u$ at (0, 0).

**For (0, $a$).** $r = -2a, s = -a, t = 0$

$\Rightarrow$ $rt - s^2$ is negative.

$\therefore$ We have neither a maximum nor a minimum of $u$ at $(a, 0)$.

Similarly, we have neither a maximum nor a minimum of $u$ at $(\alpha, 0)$.

**For $\left(\frac{1}{3}a, \frac{1}{3}a\right)$.**

$$r = -\frac{2}{3}a, s = -\frac{1}{3}a, t = -\frac{2}{3}a$$

$\Rightarrow$ $rt - s^2$ is positive.

Then, the equation (4) reduces to

$$P_{k+1}dx_{k+1}+P_{k+2}dx_{k+2}+P_{k+3}dx_{k+3}+...+P_n dx_n=0 \quad ...(5)$$

Now, let us suppose that out of $n$ variables, the $(n-k)$ variables $x_{k+1}, x_{k+2}, ..., x_n$ are independent.

Then, since $n-k$ quantities $dx_{k+1}, dx_{k+2},..., dx_n$ are independent so their coefficients must be separately zero. Hence, we have

$$P_{k+1}=0, P_{k+2}=0, ..., P_n=0$$

Thus, we have $k+n$ equations

$$P_1=0, P_2=0, ..., P_n=0$$

and $$g_1=0, g_2=0, ..., g_k=0.$$

Hence, we get $(n+k)$ equations which determine the $k$ multipliers $l_1,l_2,...,l_k$ and get the possible value of $u$.

**REMARKS**

- The Lagrange's method of undetermined multipliers is very convenient to apply. It gives the maximum and minimum values of the function without actually determining the values of the multipliers $l_1,l_2,...,l_k$.
- It does not determine the nature of stationary point, which is the only drawback of this method.

### 10.24.1 APPLICATIONS OF THE METHOD OF UNDETERMINED MULTIPLIERS

The Lagrange's method of undetermined multipliers can be applied to determine the extreme values of the given functions, it does not detemine the nature of stationary point. Now, it is more convenient to find out the extreme values of a function $F$ with the help of new function, given by

$$V=g+l_1f_1+l_2f_2+...+l_mf_m$$

and use the following method. Here, we give the method for four variables $x,y,u,v$ connected by the following two relations.

Let $F=g(x, y, u, v)$ be subjected to the conditions

$$f_1(x,y,u,v)=0 \quad ...(1)$$

and $$f_2(x,y,u,v)=0. \quad ...(2)$$

For the maxima and minima of $F$, we have

$$dF = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial v}dv = 0 \quad ...(3)$$

Now, from (1) and (2), we have $$df_1 = \frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial u}du + \frac{\partial f_1}{\partial v}dv = 0 \quad ...(4)$$

and $$df_2 = \frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_2}{\partial u}du + \frac{\partial f_2}{\partial v}dv = 0 \quad ...(5)$$

Multiplying (4) by $l_1$, (5) by $l_2$ and adding their sum to (3), we get

$$\left(\frac{\partial g}{\partial x}+l_1\frac{\partial f_1}{\partial x}+l_2\frac{\partial f_2}{\partial x}\right)dx+\left(\frac{\partial g}{\partial y}+l_1\frac{\partial f_1}{\partial y}+l_2\frac{\partial f_2}{\partial y}\right)dy$$
$$+\left(\frac{\partial g}{\partial u}+l_1\frac{\partial f_1}{\partial u}+l_2\frac{\partial f_2}{\partial u}\right)du+\left(\frac{\partial g}{\partial v}+l_1\frac{\partial f_1}{\partial v}+l_2\frac{\partial f_2}{\partial v}\right)dv=0 \quad ...(6)$$

Here, we have $l_1$ and $l_2$ are arbitrary, therefore we can choose them to satisfy the two linear equations

$$\frac{\partial g}{\partial x}+l_1\frac{\partial f_1}{\partial x}+l_2\frac{\partial f_2}{\partial x}=0 \quad ...(7)$$

and $$\frac{\partial g}{\partial y}+l_1\frac{\partial f_1}{\partial y}+l_2\frac{\partial f_2}{\partial y}=0 \quad ...(8)$$

Using (7) and (8), equation (6) reduces to

$$\left(\frac{\partial g}{\partial u}+l_1\frac{\partial f_1}{\partial u}+l_2\frac{\partial f_2}{\partial u}\right)du+\left(\frac{\partial g}{\partial v}+l_1\frac{\partial f_1}{\partial v}+l_2\frac{\partial f_2}{\partial v}\right)dv=0$$

Since, the given function contains four variables (namely $x$, $y$, $u$ and $v$) and we are given two equations of conditions, therefore, only two of the variables are independent and it is immaterial which two of the four variables are regarded as independent. Let them be $u$ and $v$ then $du$ and $dv$ are also independent, therefore, their coefficients must be zero separately. Thus

$$\frac{\partial g}{\partial u}+l_1\frac{\partial f_1}{\partial u}+l_2\frac{\partial f_2}{\partial u}=0 \quad ...(9)$$

**Example 5.** *Find the possible percentage error in computing the parallel resistance r of three resistances $r_1$, $r_2$, $r_3$ from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$. if $r_1$, $r_2$, $r_3$ are each in error by $+1.2\%$.* (UKTU–2011, 13)

**Solution.** Given that

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

$$\Rightarrow -\frac{1}{r^2}\Delta r = -\frac{1}{r_1^2}\Delta r_1 - \frac{1}{r_2^2}\Delta r_2 - \frac{1}{r_3^2}\Delta r_3$$

$$\Rightarrow \quad \frac{1}{r}\left(\frac{\Delta r}{r}\times 100\right)$$

$$= \frac{1}{r_1}\left(\frac{\Delta r_1}{r_1}\times 100\right) + \frac{1}{r_2}\left(\frac{\Delta r_2}{r_2}\times 100\right) + \frac{1}{r_3}\left(\frac{\Delta r_3}{r_3}\times 100\right)$$

$$= \frac{1}{r_1}(1.2) + \frac{1}{r_2}(1.2) + \frac{1}{r_3}(1.2)$$

$$= 1.2\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) = 1.2\left(\frac{1}{r}\right)$$

$\Rightarrow \frac{\Delta r}{r} \times 100 = 1.2$, which is the required error in $r$.

**Example 6.** (i) *In estimating the number of bricks in a pile which is measured to be (5m × 10m × 5m), the count of bricks is taken as 100 bricks per $m^3$. Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of bricks is Rs 2000 per thousand bricks.* (UPTU–2001, 07)

(ii) *In estimating the cost of a pile of bricks as 6m × 50m × 4m, the tape is stretched 1% beyond the standard length. If the count is 12 bricks is 1 $m^3$ and bricks cost Rs 1000, find the approximate error in the cost.* (UKTU–2010)

**Solution.** (i) Let length, breadth and height of the pile be $x$, $y$ and $z$ respectively.

The volume of the pile, $V = xyz$

$\Rightarrow \log V = \log x + \log y + \log z$

Differentiating, we get

$$\frac{\Delta V}{V}\times 100 = \left(\frac{\Delta x}{x}\times 100\right) + \left(\frac{\Delta y}{y}\times 100\right) + \left(\frac{\Delta z}{z}\times 100\right)$$

$$= 2 + 2 + 2 = 6$$

Therefore,

$$\Delta V = \frac{6}{100}V = \frac{6}{100}(250) = 15\text{ m}^3$$

$$[\because V = 5\times 10\times 5 = 250\text{ m}^3]$$

Hence, number of bricks in $\Delta V$

$$= 15\times 100 = 1500$$

and error in the cost

$$= 1500\times\frac{2000}{1000} = \text{Rs } 3000$$

(ii) Proceed same as in part (i), we get

$$\frac{\Delta V}{V}\times 100 = \left(\frac{\Delta x}{x}\times 100\right) + \left(\frac{\Delta y}{y}\times 100\right) + \left(\frac{\Delta z}{z}\times 100\right)$$

$$= 1 + 1 + 1 = 3$$

$$\therefore \quad \Delta V = \frac{3}{100}V = \frac{3}{100}\times 1200 = 36\text{ m}^3$$

$$[\because V = 6\times 50\times 4 = 1200\text{ m}^3]$$

$$\therefore \quad \Delta V = 36\times 12 = 432$$

and error in the cost

$$= 432\times\frac{100}{1000} = \text{Rs } 43.20$$

**Example 7.** *The period of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$. Find the maximum percentage error in T due to the possible errors upto 1% in l and 2.5% in g.* (UPTU–2004)

**Solution.** We have $T = 2\pi\sqrt{\frac{l}{g}}$

$$\Rightarrow \quad \log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$$

$$\Rightarrow \quad \frac{1}{T}\Delta T = \frac{1}{2}\frac{\Delta l}{l} - \frac{1}{2}\frac{\Delta g}{g}$$

$$\Rightarrow \quad \frac{\Delta T}{T}\times 100 = \frac{1}{2}\left[\frac{\Delta l}{l}\times 100 - \frac{\Delta g}{g}\times 100\right]$$

$$= \frac{1}{2}[1 \pm 2.5]$$

Hence, maximum error in $T$

$$= \frac{1}{2}(1 + 2.5) = 1.75\%.$$

**Example 8.** *The height h and the semi-vertical angle α of a cone are measured and from them, the total surface area of the cone including the base is calculated. If h and α are in error by small quantities δh and δα respectively. Find the corresponding error in the area. Show further that if $\alpha = \frac{\pi}{6}$ an error of $+1\%$ in h will be approximately compensated by an error of $-0.33°$ in α.* (UPTU–2009)

**Solution.** We have

Base radius, $r = h\tan\alpha$

Slant height, $l = h\sec\alpha$

Total area, $A = \pi r^2 + \pi r l$

$$= \pi r(r + l)$$

$$= \pi h\tan\alpha(h\tan\alpha + h\sec\alpha)$$

$$= \pi h^2(\tan^2\alpha + \sec\alpha\tan\alpha)$$

So, $$\delta A = \frac{\partial A}{\partial h}\delta h + \frac{\partial A}{\partial \alpha}\delta\alpha$$