The identity element for a binary operation $*$ on a set S, if it exists, is unique.

Left inverse Let $*$ be a binary operation on a set S and $e \in S$ be the identity element for $*$ on S. An element b is a left inverse of $a \in S$ if

$$b * a = e$$

Right inverse Let $*$ be a binary operation on a set S and $e \in S$ be the identity element for $*$ on S. An element c is a right inverse of $a \in S$ if

$$a * c = e$$

Inverse of an element Let $*$ be a binary operation on a set S and $e \in S$ be the identity element for $*$ on S. An element x is an inverse of an element $a \in S$ if x is both left inverse as well as right inverse of a, i.e.

$$x * a = e = a * x$$

The inverse of a is usually denoted by a^{-1}. For additive binary operation on a set S, the inverse of a is denoted by $-a$.

An element $a \in S$ is said to be invertible, if it possesses its inverse. The inverse of an invertible element is unique. The identity element is always invertible and is inverse of itself.

Algebraic Structure

A non-empty set S equipped with one or more binary operations on it is called an algebraic structure.

1.6 GROUPS

Semigroup An algebraic structure $(G, *)$ consisting of a non-void set G and a binary operation $*$ defined on G is called a semigroup if it satisfies the following axiom.

SG-1 Associativity The binary operation $*$ is associative on G

i.e. $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$

The algebraic structures $(N, +)$, $(Q, +)$ $(R, +)$, $(C, +)$ $(Z, +)$, $(Q, +)$, etc. are semigroups.

Let $P(S)$ be the power set of a set S. Then, $(P(S), \cup)$ and $(P(S), \cap)$ are semigroups.

Monoid: An algebraic structure $(G, *)$ consisting of a non-void set G and a binary operation $*$ defined on G is called a monoid if it satisfies the following axioms.

M-1 Associativity The binary operation $*$ is associative on G

i.e. $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$.

Here is the transition matrix for this problem.

$$P = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(b) Determine the matrix that has the coordinate vector$(u)_C = (-8, 3, 5, -2)$.
So, the coordinate vector is for the non-standard basis vectors. As with the previous problem we could just write down in linear combination of the vectors from C.

$$[v]_B = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -8 \\ 3 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 5 \\ 1 \end{bmatrix}$$

The coordinate for u relative to the standard basis we can write down $\boldsymbol{u}$.

$$u = \begin{bmatrix} 4 & 5 \\ -3 & 1 \end{bmatrix}$$

Example 8: Consider the two bases for R^2, $B = \{(1, -1), (0, 6)\}$ and $C = \{(2,1), (-1, 4)\}$.

(a) Find the transition matrix from C to B.

(b) Find the transition matrix from B to C.

Solution:

(a) Find the transition matrix from C to B.
The vectors from C as linear combinations of the vectors from B. Here, are those linear combinations

$$(2,1) = 2(1,-1) + \frac{1}{2}(0,6)$$

$$(-1,4) = -(1,1) + \frac{1}{2}(0,6)$$

The two coordinate matrices are

$$[(2,1)]_B = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} \qquad [(-1,4)]_B = \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$$

and the transition matrix is

4.9 VECTOR SUBSPACES

Let V be a vector space over the field F and let $W \subseteq V$. Then W is called a subspace of V, if W itself is a vector space over F w.r.t to the operation of vector addition and scalar multiplication in V.

Theorem: The necessary and sufficient conditions for a non-empty subset W of a vector space $V(F)$ to be a subspace of V are

(i) $\alpha \in W, \beta \in W \quad \Rightarrow \quad \alpha - \beta \in W$

(ii) $a \in F, \alpha \in W \quad \Rightarrow \quad a\alpha \in W$

Proof:

The conditions are necessary: If W is a subspace of V, then W is an abelian group w.r.t vector addition.

$\therefore \alpha \in W, \beta \in W \quad \Rightarrow \quad \alpha - \beta \in W$

Also W must be closed under scalar multiplication.

$\therefore$ Condition (ii) is also necessary.

The conditions are sufficient: Now suppose W is a non-empty subset of V satisfying the two given conditions.

From condition (1), we have $\alpha \in W, \alpha \in W \Rightarrow \alpha - \alpha \in W \Rightarrow 0 \in W$

Thus, the zero vector of V belongs to W and it will also be the zero vector of W.

Now, $0 \in W, \alpha \in W \Rightarrow 0 - \alpha \in W \Rightarrow -\alpha \in W$

Thus the additive inverse of each element of W is also in W.

Again $\alpha \in W, \beta \in W \Rightarrow \alpha \in W, -\beta \in W \Rightarrow \alpha - (-\beta) \in w \Rightarrow \alpha + \beta \in W$

Thus W is closed w.r.t vector addition.

Since the elements of W are also the elements of V,

$\therefore$ Vector addition will be commutative as well as associative in W.

Hence W is the abelian group under vector addition.

From condition (ii), W is closed under scalar multiplication. The remaining postulates of a vector space will hold in W since they hold in V of which W is a subset.

Hence W is subspace of V.

Example 1: The set W of ordered triads $(a_1, a_2, 0)$, where $a_1, a_2 \in F$ is a subspace of $V_3(F)$.

Solution: Let $\alpha = (a_1, a_2, 0)$ and $\beta = (b_1, b_2, 0)$ be any two elements of W and $\forall\, a_1, a_2, b_1, b_2 \in F$.

If $a, b \in F$, then we have

$$a\alpha + b\beta = a(a_1, a_2, 0) + b(b_1, b_2, 0)$$

$$= (aa_1, aa_2, 0) + (bb_1, bb_2, 0)$$

$$= (aa_1 + bb_1, aa_2 + bb_2, 0)$$

$\Rightarrow$ Set is linearly dependent

$$a(1, 2, 1) + b(3, 1, 5) + c(3, -4, 7) = (0, 0, 0)$$

$$\Rightarrow \quad a = 3, b = -2, c = 1$$

Example 5: Is the vector (2, – 5, 3) in the subspace of R^3 spanned by the vectors (1, – 3, 2), (2, – 4, – 1), (1, – 5, 7)?

Solution: Let $\alpha = (2, -5, 3)$, $\alpha_1 = (1, -3, 2)$, $\alpha_2 = (2, -4, -1)$, $\alpha_3 = (1, -5, 7)$

If α can be expressed as a linear combination of vectors $\alpha_1, \alpha_2, \alpha_3$, then it will be in the subspace of R^3 spanned by three vectors, otherwise it will not be.

Let $\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3, \qquad a_1, a_2, a_3 \in R$

$$\Rightarrow \quad (2, -5, 3) = a_1(1, -3, 2) + a_2(2, -4, -1) + a_3(1, -5, 7)$$

$$= (a_1 + 2a_2 + a_3, -3a_1 - 4a_2 - 5a_3, 2a_1 - a_2 + 7a_3)$$

$$\therefore \quad a_1 + 2a_2 + a_3 = 2 \qquad \ldots(1)$$

$$-3a_1 - 4a_2 - 5a_3 = -5 \qquad \ldots(2)$$

$$2a_1 - a_2 + 7a_3 = 3 \qquad \ldots(3)$$

$$3(1) + (2) \Rightarrow 2a_2 - 2a_3 = 1 \Rightarrow a_2 - a_3 = \frac{1}{2} \qquad \ldots(4)$$

$$2(1) - (3) \Rightarrow 5a_2 - 5a_3 = 1 \Rightarrow a_2 - a_3 = \frac{1}{5} \qquad \ldots(5)$$

We note that equations (4) and (5) are inconsistent.

$\Rightarrow$ Vector α cannot be expressed as a linear combination of $\alpha_1, \alpha_2, \alpha_3$.

$\Rightarrow$ α is not in the subspace of R^3 generated by the vectors $\alpha_1, \alpha_2, \alpha_3$.

Example 6: Are the vectors (1, 1, 2, 4), (2, –1, –5, 2) (1, –1, –4, 0) and (2, 1, 1, 6) linearly independent on R^4?

Solution:

$$(1, 1, 2, 4) = a(2, -1, -5, 2) + b(1, -1, -4, 0) + c(2, 1, 1, 6)$$

$$\Rightarrow \quad 2a + b + 2c = 1$$

$$-a - b + c = 1$$

$$-5a - 4b + c = 2$$

$$2a + 0b + 6c = 4$$

Solving $a = 2, b = -3, c = 0$

$$\therefore \quad (1, 1, 2, 4) = 2(2, -1, -5, 2) - 3(1, -1, -4, 0) + 0(2, 1, 1, 0)$$

$$1(1, 1, 2, 4) - 2(2, -1, -5, 2) + 3(1, -1, -4, 0) - 0(2, 1, 1, 6) = (0, 0, 0, 0)$$

$\Rightarrow$ Not all scalars 1, – 2, 3, 0 not all zero.

$\Rightarrow$ Given vectors are linearly dependent in R^4.