Alternatively: $\mathbf{r} = (r\cos\omega t)\,\mathbf{i} + (r\sin\omega t)\,\mathbf{j}$. ...(*i*) Therefore, $\mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt} = (-r\omega\sin\omega t)\,\mathbf{i} + (r\omega\cos\omega t)\,\mathbf{j}$,

and acc., $\mathbf{a}(t) = \dfrac{d^2\mathbf{r}}{dt^2} = (-r\omega^2\cos\omega t)\,\mathbf{i} - (r\omega^2\sin\omega t)\,\mathbf{j} = -\omega^2[(r\cos\omega t)\,\mathbf{i} + (r\sin\omega t)\,\mathbf{j}] = -\omega^2\mathbf{r}$, [from (*i*)].

EXAMPLE 1.16 Find by vector methods, the tangential and normal components of acceleration of a particle moving in a plane curve.

Solution Let $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ respectively denote unit vectors along tangent and normal at any point on the curve at any time t.

If $\mathbf{v}$ denotes velocity at P along the tangent then $\mathbf{v} = v\hat{\mathbf{T}}$ where $v = |\mathbf{v}|$.

Acceleration along tangent $= \dfrac{d\mathbf{v}}{dt} = \dfrac{dv}{dt}\hat{\mathbf{T}} + v\dfrac{d\hat{\mathbf{T}}}{dt} = \dfrac{dv}{dt}\hat{\mathbf{T}} + v\dfrac{d\hat{\mathbf{T}}}{d\psi}\cdot\dfrac{d\psi}{ds}\cdot\dfrac{ds}{dt}$

$$= \frac{dv}{dt}\hat{\mathbf{T}} + v^2\frac{1}{\rho}\frac{d\hat{\mathbf{T}}}{d\psi} \qquad ....(1)$$

$\left(\because \dfrac{d\psi}{ds} = \dfrac{1}{\rho}\right.$, where ρ is radius of circle and $\left.\dfrac{ds}{dt} = v.\right)$

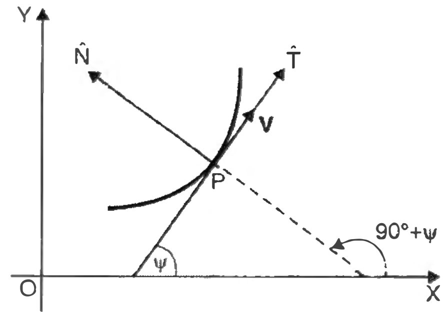

Fig. 1.7

Now $\hat{\mathbf{T}} = \cos\psi\,\mathbf{i} + \sin\psi\,\mathbf{j}$, and $\hat{\mathbf{N}} = -\sin\psi\,\mathbf{i} + \cos\psi\,\mathbf{j}$.

$$\therefore \quad \frac{d\hat{\mathbf{T}}}{d\psi} = -\sin\psi\,\mathbf{i} + \cos\psi\,\mathbf{j} = \hat{\mathbf{N}}.$$

Hence from (1) we have $\dfrac{d\mathbf{v}}{dt} = \dfrac{dv}{dt}\hat{\mathbf{T}} + \dfrac{v^2}{\rho}\hat{\mathbf{N}}$.

Thus tangential and normal components of acceleration respectively are $\dfrac{dv}{dt}$ and $\dfrac{v^2}{\rho}$.

EXAMPLE 1.17 Find the radial and transverse components of velocity and acceleration of a particle describing a plane curve. (*Kurukshetra, 2006; Rajasthan, 2006*)

Solution Let $P(r, \theta)$ be the position of a moving particle at any time t along a curve $r = f(\theta)$. Let $\hat{\mathbf{R}}$ and $\hat{\mathbf{S}}$ respectively denote unit vectors in the radial and transverse directions, that is, along and perpendicular to OP.

Then $\hat{\mathbf{R}} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$,

and $\hat{\mathbf{S}} = \cos\left(\dfrac{\pi}{2}+\theta\right)\mathbf{i} + \sin\left(\dfrac{\pi}{2}+\theta\right)\mathbf{j} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$.

Therefore, $\dfrac{d\hat{\mathbf{R}}}{dt} = (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})\dfrac{d\theta}{dt} = \hat{\mathbf{S}}\dfrac{d\theta}{dt}$ ...(*i*)

and $\dfrac{d\hat{\mathbf{S}}}{dt} = (-\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j})\dfrac{d\theta}{dt} = -\hat{\mathbf{R}}\dfrac{d\theta}{dt}$. ...(*ii*)

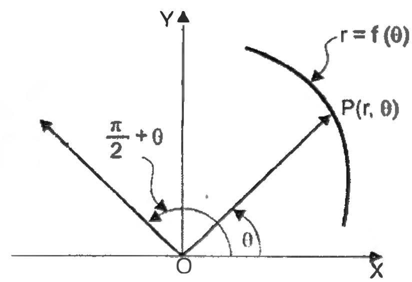

Fig. 1.8

Now $\mathbf{R}$, the position vector of P is given by $\mathbf{R} = r\hat{\mathbf{R}}$ [where $r = |\mathbf{R}|$].

Therefore, $\dfrac{d\mathbf{R}}{dt} = \dfrac{dr}{dt}\hat{\mathbf{R}} + r\dfrac{d\hat{\mathbf{R}}}{dt} = \dfrac{dr}{dt}\hat{\mathbf{R}} + r\dfrac{d\theta}{dt}\hat{\mathbf{S}}$. ...[By (*i*)]

Hence *radial component* of velocity is dr/dt and *transverse component* is $r(d\theta/dt)$.

As $\dfrac{d\mathbf{R}}{dt} = \dfrac{dr}{dt}\hat{\mathbf{R}} + r\dfrac{d\theta}{dt}\hat{\mathbf{S}}$.

Therefore, $\nabla \cdot \nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(6x^2y^2z^4\mathbf{i} + 4x^3yz^4\mathbf{j} + 8x^3y^2z^3\mathbf{k})$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2 = F(x, y, z) \text{ say.}$$

$$\nabla F(x, y, z) = \nabla(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2)$$

$$= \mathbf{i}\frac{\partial}{\partial x}(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2) + \mathbf{j}\frac{\partial}{\partial y}(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2) + \mathbf{k}\frac{\partial}{\partial z}(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2)$$

$$= (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\mathbf{i} + (24xyz^4 + 48x^3yz^2)\mathbf{j} + (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\mathbf{k}$$

and $\nabla F(1, -2, 1) = (48 + 12 + 288)\mathbf{i} + (-48 - 96)\mathbf{j} + (192 + 16 + 192)\mathbf{k} = 348\,\mathbf{i} - 144\,\mathbf{j} + 400\,\mathbf{k}$.

Therefore, $D_{\mathbf{b}}F(1, -2, 1) = \nabla F \cdot \hat{\mathbf{b}} = (348\mathbf{i} - 144\mathbf{j} + 400\mathbf{k})\cdot\frac{(\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})}{\sqrt{21}}$

$$= \frac{(348)(1) + (-144)(-4) + (400)(2)}{\sqrt{21}} = \frac{1724}{\sqrt{21}}.$$

EXAMPLE 1.35 Find the directional derivative of $f(x, y, z) = x^2yz^3$ along the curve $x = e^{-u}$, $y = 2\sin u + 1$, $z = u - \cos u$ at the point P where $u = 0$.

Solution The point P corresponding to $u = 0$ is $(1, 1, -1)$.

$$\nabla f(x, y, z) = 2xyz^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}, \quad \text{and} \quad \nabla f(1, 1, -1) = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

The position vector $\mathbf{r}$ of any point on the given curve is given by

$$\mathbf{r} = e^{-u}\,\mathbf{i} + (2\sin u + 1)\mathbf{j} + (u - \cos u)\mathbf{k}.$$

A tangent vector to the curve is

$$\mathbf{r}'(u) = \frac{d\mathbf{r}}{du} = \frac{d}{du}\{e^{-u}\mathbf{i} + (2\sin u + 1)\mathbf{j} + (u - \cos u)\mathbf{k}\}$$

$= -e^{-u}\,\mathbf{i} + 2\cos u\,\mathbf{j} + (1 + \sin u)\mathbf{k}$. Therefore, $(d\mathbf{r}/du)_{u=0}$, that is, $\mathbf{r}'(0) = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

The unit tangent vector is given by $\hat{\mathbf{b}} = \frac{-\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4 + 1}} = \frac{-\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{6}}$.

Therefore, $D_{\mathbf{b}}f(1, 1, -1) = \nabla f \cdot \hat{\mathbf{b}} = (-2\mathbf{i} - \mathbf{j} + 3\mathbf{k})\cdot\left(\frac{-\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{6}}\right) = \frac{1}{\sqrt{6}}(2 - 2 + 3) = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$.

1.13 PROPERTIES OF GRADIENT

If f and g are differentiable scalar functions of position (x, y, z), then

(*i*) $\nabla(kf) = k\,\nabla f$ (any number k), (*ii*) $\nabla(f \pm g) = \nabla f \pm \nabla g$,

(*iii*) $\nabla(c_1 f + c_2 g) = c_1\,\nabla f + c_2\,\nabla g$, c_1, c_2 arbitrary constants, (*iv*) $\nabla(fg) = f\,\nabla g + g\,\nabla f$,

(*v*) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$, $g \neq 0$.

The proof of properties (*ii*), (*iv*) and (*v*) is given below.

(*ii*) $\nabla(f \pm g) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(f \pm g) = \mathbf{i}\frac{\partial}{\partial x}(f \pm g) + \mathbf{j}\frac{\partial}{\partial y}(f \pm g) + \mathbf{k}\frac{\partial}{\partial z}(f \pm g)$

$$= \left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right) \pm \left(\mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j}\frac{\partial g}{\partial y} + \mathbf{k}\frac{\partial g}{\partial z}\right) = \nabla f \pm \nabla g.$$

$$= \mathbf{u}\,(\nabla \cdot \mathbf{v}) - (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \cdot \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\mathbf{v}$$

$$+ (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \cdot \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\mathbf{u} - \mathbf{v}\,(\nabla \cdot \mathbf{u})$$

$$= \mathbf{u}\,(\nabla \cdot \mathbf{v}) - (\mathbf{u} \cdot \nabla)\,\mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{u} - \mathbf{v}\,(\nabla \cdot \mathbf{u}).$$

Rearranging the terms on R.H.S., we obtain

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}\,(\nabla \cdot \mathbf{v}) - \mathbf{v}\,(\nabla \cdot \mathbf{u}) + (\mathbf{v} \cdot \nabla)\,\mathbf{u} - (\mathbf{u} \cdot \nabla)\,\mathbf{v}$$

or $$\text{Curl}\,(\mathbf{u} \times \mathbf{v}) = \mathbf{u}\,(\text{div}\,\mathbf{v}) - \mathbf{v}\,(\text{div}\,\mathbf{u}) + (\mathbf{v} \cdot \nabla)\,\mathbf{u} - (\mathbf{u} \cdot \nabla)\,\mathbf{v}.$$

EXAMPLE 1.45 Prove the following identities:

(*i*) curl (grad f) = **0** or $\nabla \times (\nabla f) = \mathbf{0}$, (*PTU 2000*)

(*ii*) div (curl **v**) = 0 or $\nabla \cdot (\nabla \times \mathbf{v}) = 0$, (*MDU 2003*)

(*iii*) curl (curl **v**) *i.e.*, $\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2\mathbf{v}$,

or grad (div **v**) = $\nabla \times (\nabla \times \mathbf{v}) + \nabla^2\mathbf{v}$ where **v**, a differentiable vector field.

(*PTU 2006; AMIETE Dec 2004; UPTU 2003; MDU 2002*)

Solution (*i*) From the definition, we have

$$\nabla \times (\nabla f) = \nabla \times \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial \mathbf{i}}{\partial z}\mathbf{k}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]\mathbf{k}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k} = \mathbf{0}.$$

(*ii*) We have **v**, a differentiable vector field = $v_1\mathbf{i} + v_2\,\mathbf{j} + v_3\mathbf{k}$.

From the definition $$\text{div}\,(\text{curl}\,\mathbf{v}) = \nabla \cdot (\nabla \times \mathbf{v}) = \nabla \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \nabla \cdot \left[\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} + \frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y} = 0.$$

(*iii*) $$\text{curl}\,(\text{curl}\,\mathbf{v}) = \nabla \times (\nabla \times \mathbf{v}) = \left(\sum \mathbf{i}\frac{\partial}{\partial x}\right) \times \left[\sum \mathbf{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\right]$$

$$= \sum \mathbf{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\right] = \sum \mathbf{i}\left[\frac{\partial^2 v_2}{\partial y \partial x} + \frac{\partial^2 v_3}{\partial z \partial x} - \left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right)\right]$$

$$= \sum \mathbf{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) - \left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right)\right] = \sum \mathbf{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) - \left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right)\right]$$

EXAMPLE 1.56 Evaluate $\int_C (x + y^2 + yz)\, ds$, where C is the curve defined by $y = 2x$, $z = 2$ from $(1, 2, 2)$ to $(3, 6, 2)$.

Solution Let $x = t$. Then $y = 2t$, $z = 2$. Therefore, the curve C is represented by $x = t$, $y = 2t$, $z = 2$, $1 \le t \le 3$.

We have $\frac{ds}{dt} = \sqrt{5}$. Therefore, $\int_C (x + y^2 + yz) \frac{ds}{dt} dt = \sqrt{5} \int_1^3 (t + 4t^2 + 4t)\, dt = \sqrt{5}\left[5\frac{t^2}{2} + \frac{4t^3}{3}\right]_1^3$

$$= \sqrt{5}\left(\frac{45}{2} + 36 - \frac{5}{2} - \frac{4}{3}\right) = \frac{164}{3}\sqrt{5}.$$

Line Integral of Vector Fields

Let C be a smooth curve. Let $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$ be a vector field, that is, continuous on C. We define $d\mathbf{r} = \mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz$.

Then, the line integral of $\mathbf{F}$ over C is defined by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_1\, dx + F_2\, dy + F_3\, dz. \qquad \text{...(1.27)}$$

Let the parametric representation of simple curve C be

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad t_0 \le t \le t_1.$$

Then $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} F[x(t), y(t), z(t)] \cdot \frac{dr}{dt}\, dt. \qquad \text{...(1.28)}$$

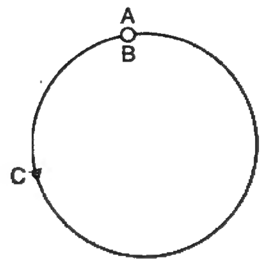

Fig. 1.14

If the path of integration C in eq. (1.28) is a closed curve. (Ref. Fig. 1.14), then instead of $\int_C$ we write $\oint_C$.

If the curve C is piecewise smooth Fig. (1.15) containing the arcs $C_1, C_2, \ldots C_n$ joined end to end, we define a line integral along a piecewise smooth curve C to be the sum of the integrals along the sections.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \ldots\ldots + \int_{C_n} \mathbf{F} \cdot d\mathbf{r}. \qquad \text{...(1.29)}$$

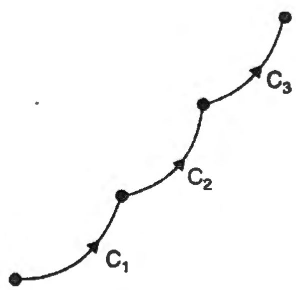

Fig. 1.15

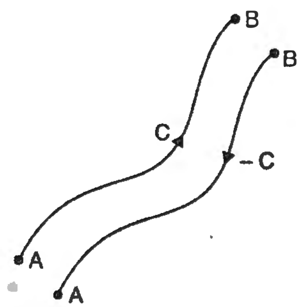

Fig. 1.16

Remark 10

Reversal of the path of integration changes the sign of the line integral. (Fig. 1.16)

EXAMPLE 1.57 If $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C in the xy plane consisting of:

(*i*) the straight line joining (0, 0) to (1, 1),
(*ii*) the parabola $y^2 = x$ joining the same points,
(*iii*) the straight lines from (0, 0) to (1, 0), then to (1, 1).

Solution Since $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$.