Distance between two complex numbers: The distance between two complex numbers z_1 and z_2 is given by

$$d = |z_2 - z_1| = |(x_2 - x_1) + i(y_2 - y_1)| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Remark 1

We cannot compare two complex numbers that is, for any two complex numbers z_1, z_2 we cannot write $z_1 > z_2$ or $z_1 < z_2$. However, we can compare the magnitudes of two complex numbers.

EXAMPLE 1.1 A rectangle is constructed in the complex plane with its sides parallel to the axes and its centre situated at the origin. If one of the vertices of the rectangle is $1 + i\sqrt{3}$, find the complex numbers representing the other three vertices. Find also the area of the rectangle.

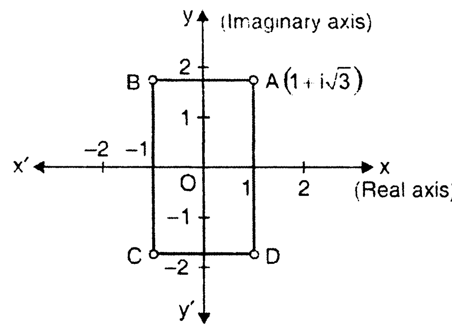

Fig. 1.6

Solution Since A represents $1 + i\sqrt{3}$, its co-ordinates are $(1, \sqrt{3})$.

From symmetry, the co-ordinates of B, C and D are respectively $(-1, \sqrt{3}), (-1, -\sqrt{3})$ and $(1, -\sqrt{3})$.

Therefore, complex numbers representing B, C, D are $(-1 + i\sqrt{3})$, $(-1 - i\sqrt{3})$ and $(1 - i\sqrt{3})$ respectively.

The area of the rectangle $ABCD = (2)(2\sqrt{3}) = 4\sqrt{3}$ sq units.

EXAMPLE 1.2 Find real and imaginary parts of $(z^3 - z)$.

Solution Let $z = x + iy$. Therefore, $z^3 - z = (x + iy)^3 - (x + iy) = x^3 + i^3y^3 + 3ixy(x + iy) - x - iy$

$$= x^3 + 3ix^2y - 3xy^2 - iy^3 - x - iy$$

$$= (x^3 - 3xy^2 - x) + i(-y^3 + 3x^2y - y).$$

Therefore, $\text{Re}(z^3 - z) = x^3 - 3xy^2 - x.$ $\quad \text{Im}(z^3 - z) = -y^3 + 3x^2y - y.$

EXAMPLE 1.3 Express the function in the form $u(x, y) + iv(x, y)$, where u and v are real. $f(z) = z^3 + 2iz$.

Solution We have

$$f(z) = z^3 + 2iz = (x + iy)^3 + 2i(x + iy)$$

$$= x^3 + i^3y^3 + 3ixy(x + iy) + 2ix + 2i^2y$$

$$= x^3 - iy^3 + 3ix^2y - 3xy^2 + 2ix - 2y$$

$$= (x^3 - 3xy^2 - 2y) + i(3x^2y - y^3 + 2x).$$

Hence, we obtain $u(x, y) = x^3 - 3xy^2 - 2y$, $v(x, y) = 3x^2y - y^3 + 2x$.

EXAMPLE 1.4 Express the following complex numbers in the form $x + iy$, where x and y are real.

(*i*) $\dfrac{(1+i)(2-i)}{3+i}$, (*ii*) $\dfrac{2-3i}{4+6i}$, (*iii*) $\dfrac{\overline{3-2i}}{(4-5i)(2+i)}$, (*iv*) $\dfrac{(1-2i)^3}{(1+i)(2-i)(1-3i)}$, (*v*) $\dfrac{i^4+i^9+i^{16}}{2-i^5+i^{10}-i^{15}}$.

Solution (*i*) The numerator $= (1 + i)(2 - i) = 2 + i - i^2 = 3 + i$.

Therefore, $\dfrac{(1+i)(2-i)}{3+i} = \dfrac{3+i}{3+i} = 1 = 1 + 0i.$

(*ii*) Multiplying the numerator and the denominator by the conjugate of the complex number in the denominator, the given expression becomes

$$\frac{(2-3i)(4-6i)}{(4+6i)(4-6i)} = \frac{8-24i+18i^2}{16-36i^2} = \frac{-10-24i}{52} = \frac{-5}{26} - \frac{6}{13}i.$$

(*iii*) $$\frac{\overline{3-2i}}{(4-5i)(2+i)} = \frac{3+2i}{8-6i-5i^2} = \frac{3+2i}{13-6i} = \frac{(3+2i)(13+6i)}{(13-6i)(13+6i)} = \frac{39+44i+12i^2}{169-36i^2} = \frac{27+44i}{205} = \frac{27}{205} + \frac{44}{205}i.$$

(*iv*) The given expression $= \dfrac{1-6i+12i^2-8i^3}{(3+i)(1-3i)} = \dfrac{1-6i-12+8i}{6-8i} = \dfrac{-11+2i}{6-8i} = \dfrac{(-11+2i)(6+8i)}{(6-8i)(6+8i)}$

(*i*) The equations of line AB are $x/2 = y = z/3 = t$ (say).
Therefore, $x = 2t$, $y = t$ and $z = 3t$ are the parametric equations of the line AB.
The points (0, 0, 0) to (2, 1, 3) correspond to $t = 0$ and $t = 1$ respectively.

$$\text{Work done} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 3(2t)^2 d(2t) + [2(2t)(3t) - t]dt + 3t\, d(3t)$$

$$= \int_0^1 24t^2 dt + (12t^2 - t)dt + 9t\, dt = \int_0^1 (36t^2 + 8t)dt = 16.$$

(*ii*) $$\text{Work done} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 3(2t^2)^2 d(2t^2) + [2(2t^2)(4t^2 - t) - t]dt + (4t^2 - t)\,d\,(4t^2 - t)$$

$$= \int_0^1 48t^5 dt + (16t^4 - 4t^3 - t)dt + (32t^3 - 12t^2 + t)dt$$

$$= \int_0^1 (48t^5 + 16t^4 + 28t^3 - 12t^2)dt = \left[8t^6 + \frac{16}{5}t^5 + 7t^4 - 4t^3\right]_0^1 = 14.2.$$

(*iii*) Let $x = t$ in $x^2 = 4y$, $3x^3 = 8z$. Then the parametric equations of C are $x = t,\ y = t^2/4,\ z = 3t^3/8$.
Obviously t varies from 0 to 2. Hence the work done $= \int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_0^2 3t^2 dt + \left[2(t)\left(\frac{3t^3}{8}\right) - \frac{t^2}{4}\right] d\left(\frac{t^2}{4}\right) + \frac{3t^3}{8}\, d\left(\frac{3t^3}{8}\right)$$

$$= \int_0^2 \left(3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right)dt = \left[t^3 - \frac{t^4}{32} + \frac{17t^6}{128}\right]_0^2 = 16.$$

EXAMPLE 8.64 Find the work done by the force $\mathbf{F} = x\,\mathbf{i} - z\,\mathbf{j} + 2y\,\mathbf{k}$ in the displacement along the closed path C consisting of the segments C_1, C_2 and C_3 where on C_1, $0 \le x \le 1$, $y = x$, $z = 0$, on C_2, $0 \le z \le 1$, $x = 1$, $y = 1$ on C_3, $1 \ge x \ge 0$, $y = z = x$.

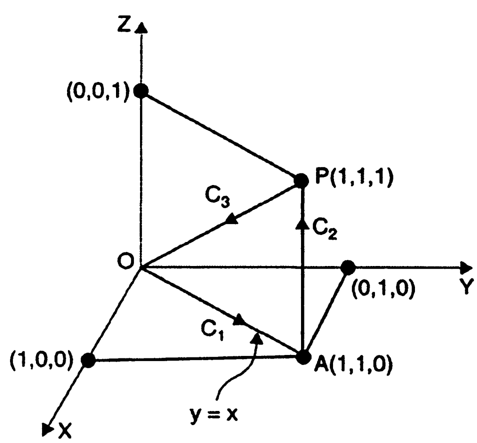

Fig. 8.20

Solution Total work done $= \oint_C \mathbf{F}\cdot d\mathbf{r}$

$$= \oint_C (x\mathbf{i} - z\,\mathbf{j} + 2y\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C x\,dx - z\,dy + 2y\,dz.$$

The closed path C consisting of segments C_1, C_2 and C_3 is shown in the Fig. 8.20.

Let W_1, W_2, W_3 be the work done in displacement along C_1, C_2 and C_3 respectively.

Since on C_1, $0 \le x \le 1$, $y = x$, $z = 0$, $dy = dx$, $dz = 0$.

Therefore, $$W_1 = \int_{C_1} x\,dx = \int_0^1 x\,dx = 1/2.$$

On C_2, $0 \le z \le 1$, $x = 1$, $y = 1$, $dx = 0$, $dy = 0$. Therefore, $W_2 = \int_{C_2} 2dz = 2\int_0^1 dz = 2.$

On C_3, $1 \ge x \ge 0$, $y = z = x$, $dy = dz = dx$. Therefore, $W_3 = \int_{C_3} x\,dx - x\,dx + 2x\,dx = 2\int_1^0 x\,dx = -1.$

Thus, total work done $= W_1 + W_2 + W_3 = (1/2) + 2 - 1 = 3/2.$

EXAMPLE 8.65 Find the constant a so that the vector field $\mathbf{v} = (axy - z^3)\mathbf{i} + (a - 2)\,x^2\mathbf{j} + (1 - a)xz^2\mathbf{k}$ is conservative. Calculate its scalar potential and the work done in moving a particle from $P(1, 2, -3)$ to $Q(1, -4, 2)$ in the field.

Therefore,

$$\oint_C (xy+y^2)\,dx + x^2 dy = (19/20) - 1 = -(1/20). \qquad ...(i)$$

Also $$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dydx = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2)\right] dydx = \iint_R (x-2y)\,dydx$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (x-2y)\,dydx = \int_0^1 [xy-y^2]_{x^2}^{x}\,dx = \int_0^1 (x^4-x^3)\,dx = \left[\frac{x^5}{5}-\frac{x^4}{4}\right]_0^1 = -\frac{1}{20}.$$

This agrees with the result obtained in (*i*). Hence, the Green's theorem is verified.

EXAMPLE 8.72 Apply Green's theorem to evaluate $\oint_C (y-\sin x)\,dx + \cos x\,dy$, where C is the triangle enclosed by the lines $y=0$, $x=\pi/2$ and $y=2x/\pi$.

(*AMIETE, June 2013, Dec. 2009; MDU 2007; PTU 2006; AMIE-2005; JNTU 2005*)

Solution We have $f(x, y) = y - \sin x$, $g(x, y) = \cos x$. Using the Green's theorem, we obtain

$$\oint_C (y-\sin x)\,dx + \cos x\,dy = \iint_R \left[\frac{\partial}{\partial x}(\cos x) - \frac{\partial}{\partial y}(y-\sin x)\right] dydx$$

$$= \iint_R (-\sin x - 1)\,dydx = -\int_{x=0}^{\pi/2}\left[\int_{y=0}^{2x/\pi}(1+\sin x)\,dy\right]dx$$

$$= -\int_0^{\pi/2} [(1+\sin x)\,y]_0^{2x/\pi}\,dx = -\int_0^{\pi/2}(1+\sin x)\frac{2x}{\pi}\,dx$$

$$= \frac{-2}{\pi}\int_0^{\pi/2}(x + x\sin x)\,dx = \frac{-2}{\pi}\left[\frac{x^2}{2} + \{x(-\cos x) + \sin x\}\right]_0^{\pi/2}$$

$$= -\left(\frac{\pi}{4} + \frac{2}{\pi}\right), \text{ the required value.}$$

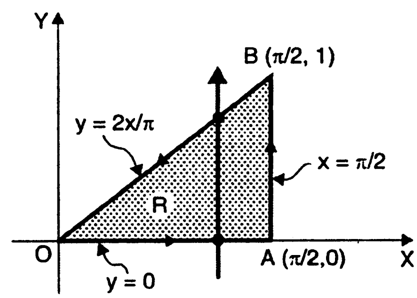

Fig. 8.25

EXAMPLE 8.73 Verify the Green's theorem for $f(x, y) = e^{-x}\sin y$, $g(x, y) = e^{-x}\cos y$ and C is the square with vertices at (0, 0), ($\pi/2$, 0), ($\pi/2$, $\pi/2$), (0, $\pi/2$). (*AMIETE-June 2010, Dec. 2005; AMIE S-2008*)

Solution We can write the line integral as

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \left[\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right](f(x,y)\,dx + g(x,y)\,dy)$$

where C_1, C_2, C_3 and C_4 are the boundary lines shown in Fig. 8.26. We have along C_1: $y = 0$, $0 \le x \le \pi/2$ and

$\int_{C_1} e^{-x}(\sin y\,dx + \cos y\,dy) = 0$, along C_2: $x = \pi/2$, $0 \le y \le \pi/2$

and $\int_{C_2} e^{-x}(\sin y\,dx + \cos y\,dy) = \int_0^{\pi/2} e^{-\pi/2}\cos y\,dy = e^{-\pi/2}$,

along C_3: $y = \pi/2$, $\pi/2 \le x \le 0$ and $\int_{C_3} e^{-x}(\sin y\,dx + \cos y\,dy)$

$$= \int_{\pi/2}^{0} e^{-x}dx = [-e^{-x}]_{\pi/2}^{0} = [e^{-x}]_0^{\pi/2} = e^{-\pi/2} - 1,$$

along C_4: $x = 0$, $\pi/2 \le y \le 0$ and

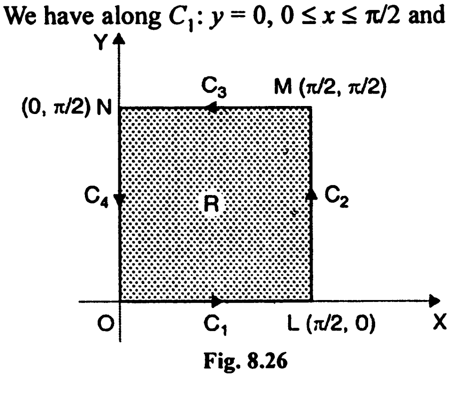

Fig. 8.26

A domain D is called *simply connected* if every simple closed contour C lying entirely in D can be shrunk, or contracted to a point without leaving D. In other words, in a simply connected domain every simple closed contour C lying entirely within it encloses only points of the domain D. A domain which is not simply connected is called *multiply connected* domain. For example, the domain bounded by a circle or a square is simply connected domain. Fig. 8.28(c).

A plain sheet of paper from which some interior parts are removed is a multiply connected domain Fig. 8.28(d). Another simple example of a multiply connected domain is the annulus $r < |z| < R$, that is, the domain between two concentric circles $|z| = R$ and $|z| = r$ cannot be shrunk to a point. Fig. 8.28(e). In plain terms a *simply connected* domain is one which has no "holes" in it. A domain with one "hole" is called *doubly connected* and a domain with two "holes" is called *triply connected* and so on.

EXERCISE 8.5

1. Apply Green's theorem to evaluate line integral $\oint_C [\sin y\, dx + x(1+\cos y)\, dy]$ over a circular path C, $x^2 + y^2 = a^2$, $z = 0$. *(AMIETE, Dec. 2011)*

   [**Hint:** In this problem $f(x, y) = \sin y$, $g(x, y) = x(1 + \cos y)$. $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1 + \cos y - \cos y = 1$.

   By Green's theorem, the given line integral is

   $$\iint_R 1\, dy\, dx = \iint_{x^2+y^2=a^2} .\, dx dy = \text{area of circle of radius } a = \pi a^2 \Big].$$

2. Using Green's theorem to evaluate $\oint_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$, where C is the boundary in the xy-plane of the area enclosed by the x-axis and the semi-circle $x^2 + y^2 = 1$ in the upper half xy-plane. *(AMIETE, Dec. 2012; AMIE S-2010)* **Ans.** 4/3.

3. Verify Green's theorem in plane for $\oint_C (x^2 - 2xy)dx + (x^2 y + 3)dy$ where C is the boundary of the region defined by $y^2 = 8x$ and $x = 2$, using the Green's theorem.

   [**Hint:** Refer Fig. 8.29

   Line integral $= \oint_C = \int_{OA} + \int_{ADB} + \int_{BO}$

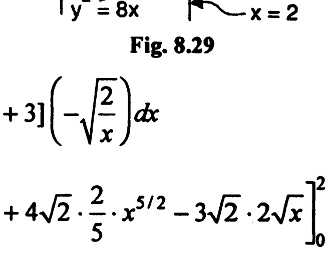

Fig. 8.29

Along OA: $y = -2\sqrt{2}\sqrt{x}$ so $dy = -\sqrt{\frac{2}{x}}dx$.

$$\int_{OA} (x^2 - 2xy)dx + (x^2 y + 3)dy = \int_0^2 [x^2 - 2x(-2\sqrt{2}\sqrt{x})]\, dx + [x^2(-2\sqrt{2}\sqrt{x}) + 3]\left(-\sqrt{\frac{2}{x}}\right)dx$$

$$= \int_0^2 (5x^2 + 4\sqrt{2}\, x^{3/2} - 3\sqrt{2}\, x^{-1/2})dx = \left[\frac{5}{3}x^3 + 4\sqrt{2}\cdot\frac{2}{5}\cdot x^{5/2} - 3\sqrt{2}\cdot 2\sqrt{x}\right]_0^2$$

$$= \left(\frac{40}{3} + \frac{64}{5} - 12\right).$$