Solution: We know that the Jacobi series are

$$J_0(x)+2J_2(x)\cos 2\theta+2J_4(x)\cos 4\theta+\ldots=\cos(x\sin\theta) \quad \ldots(1)$$

and $$2J_1(x)\sin\theta+2J_3(x)\sin 3\theta+\ldots=\sin(x\sin\theta). \quad \ldots(2)$$

Squaring (1) and (2) and integrating w.r.t. 'θ' between the limits 0 to π and using the integrals, if m, n are integers

$$\int_0^\pi \cos^2 n\theta\, d\theta=\int_0^\pi \sin^2 n\theta\, d\theta=\frac{\pi}{2}$$

and $$\int_0^\pi \cos m\theta\cos n\theta\, d\theta=\int_0^\pi \sin m\theta\sin n\theta\, d\theta=0,\ m\neq n.$$

We get $\pi[J_0(x)]^2+2\pi[J_2(x)]^2+2\pi[J_4(x)]^2+\ldots$

$$=\int_0^\pi \cos^2(x\sin\theta)d\theta \quad \ldots(3)$$

and $$2\pi[J_1(x)]^2+2\pi[J_3(x)]^2+\ldots=\int_0^\pi \sin^2(x\sin\theta)d\theta. \quad \ldots(4)$$

Adding (3) and (4), we have

$$\pi\left\{[J_0(x)]^2+2[J_1(x)]^2+2[J_2(x)]^2+\ldots\right\}=\int_0^\pi d\theta=\pi.$$

Hence, we have the required result.

Example 12: If $a>0$, prove that $\int_0^\infty e^{-ax}J_0(bx)dx=\frac{1}{\sqrt{a^2+b^2}}$.

Solution: We know that $J_0(x)=\frac{1}{\pi}\int_0^\pi \cos(x\sin\phi)d\phi$

$$\Rightarrow \quad J_0(bx)=\frac{1}{\pi}\int_0^\pi \cos(bx\sin\phi)d\phi.$$

Therefore $\int_0^\infty e^{-ax}J_0(bx)dx=\int_0^\infty e^{-ax}\left\{\frac{1}{\pi}\int_0^\pi \cos(bx\sin\phi)d\phi\right\}dx$

$$=\frac{1}{\pi}\int_0^\infty\left\{\int_0^\pi e^{-ax}\cos(bx\sin\phi)d\phi\right\}dx$$

$$=\frac{1}{\pi}\int_0^\pi\left\{\int_0^\infty e^{-ax}\cos(bx\sin\phi)dx\right\}d\phi$$

[on changing order of integration]

$$=\frac{1}{\pi}\int_0^\pi\left\{\int_0^\infty e^{-ax}\frac{e^{ibx\sin\phi}+e^{-ibx\sin\phi}}{2}dx\right\}d\phi$$

$$= x^0 \left[c_1 J_n(x^2) + c_2 J_{-n}(x^2) \right]$$

$$= c_1 J_n(x^2) + c_2 J_{-n}(x^2)$$

When n is an integer, the solution of (1) is

$$y = x^\alpha \left[c_1 J_m(\beta x^\gamma) + c_2 Y_m(\beta x^\gamma) \right]$$

$$= x^0 \left[c_1 J_n(x^2) + c_2 Y_n(x^2) \right] = c_1 J_n(x^2) + c_2 Y_n(x^2).$$

Example 26: Solve the differential equation $4y'' + 9xy = 0$ in terms of Bessel's functions.

Solution: The given equation can be written as

$$x^2 y'' + \frac{9}{4} x^3 y = 0 \qquad \text{...(1)}$$

Comparing with the general form (6) of section 4.6.7, we get

$$1 - 2\alpha = 0, \; \beta^2 \gamma^2 = \frac{9}{4}, \; 2\gamma = 3 \text{ and } \alpha^2 - n^2 \gamma^2 = 0.$$

On solving these equations, we get

$$\alpha = \frac{1}{2}, \gamma = \frac{3}{2}, \beta^2 . \frac{9}{4} = \frac{9}{4} \Rightarrow \beta = 1$$

and $\left(\frac{1}{4}\right) - n^2 \left(\frac{9}{4}\right) = 0 \Rightarrow n^2 = \frac{1}{9} \Rightarrow n = \frac{1}{3},$

Since n is not an integer, the solution of equation (1) is

$$y = x^\alpha \left[c_1 J_n(\beta x^\gamma) + c_2 J_{-n}(\beta x^\gamma) \right]$$

$$\Rightarrow \quad y = x^{1/2} \left[c_1 J_{1/3}(x^{3/2}) + c_2 J_{-1/3}(x^{3/2}) \right].$$

Example 27: Solve the differential equation $y'' + \frac{y'}{x} + \left(\frac{8}{x} - \frac{1}{x^2}\right) y = 0$ in terms of Bessel's functions.

Solution: The given equation is

$$x^2 y'' + xy' + (8x - 1) y = 0 \qquad \text{...(1)}$$

Comparing (1) with the general form

$$x^2 y'' + (1 - 2\alpha) x y' + \left[\beta^2 \gamma^2 x^{2\gamma} + (\alpha^2 - n^2 \gamma^2) \right] y = 0 \qquad \text{...(2)}$$

where α, β, γ and n are constants, we get

$$1 - 2\alpha = 1, \; \beta^2 \gamma^2 = 8, \; 2\gamma = 1 \text{ and } \alpha^2 - n^2 \gamma^2 = -1$$

Solving these, we get

$$\alpha = 0, \; \gamma = \frac{1}{2}, \; \frac{\beta^2}{4} = 8 \Rightarrow \beta^2 = 32 \Rightarrow \beta = 4\sqrt{2},$$

Now for $y=1$, we have $p=2xy\phi'_1(x^2y)+y^2\phi'_2(xy^2)=2x$ ...(9)

And $q=x^2\phi'_1(x^2y)+2xy\phi'_2(xy^2)=-2y$ ...(10)

Solving (9) and (10), we get $\phi'_1(x^2y)=\dfrac{4}{3y}+\dfrac{2y}{3x^2}$ ...(11)

And $\phi'_2(xy^2)=-\dfrac{2x}{3y^2}-\dfrac{4}{3x}$ ...(12)

Putting $y=1$ in (11), we have

$$\phi'_1(x^2)=\frac{4}{3}+\frac{2}{3x^2} \text{ or } 2x\phi'_1(x^2)=\frac{8}{3}x+\frac{2}{3}\frac{2x}{x^2}.$$

Integrating, we get

$$\phi_1(x^2)=\frac{4}{3}x^2+\frac{2}{3}\log x^2.$$

Substituting x^2y for x^2 $(\because y=1)$, we have

$$\phi_1(x^2y)=\frac{4}{3}x^2y+\frac{2}{3}\log(x^2y) \quad ...(13)$$

Also putting $y=1$ in (12), we have $\phi'_2(x)=-\dfrac{2}{3}x-\dfrac{4}{3x}$.

Integrating, we obtain

$$\phi_2(x)=-\frac{1}{3}x^2-\frac{4}{3}\log x.$$

Substituting xy^2 for x $(\because y=1)$, we have

$$\phi_2(xy^2)=-\frac{1}{3}(xy^2)^2-\frac{4}{3}\log(xy^2) \quad ...(14)$$

Using (13) and (14) in (7), we have

$$z=\frac{4}{3}x^2y+\frac{2}{3}\log(x^2y)-\frac{1}{3}x^2y^4-\frac{4}{3}\log(xy^2)+c$$

$$\Rightarrow \quad z=\frac{4}{3}x^2y-\frac{1}{3}x^2y^4+\frac{2}{3}\log\{(x^2y)/(xy^2)^2\}+c$$

$$\Rightarrow \quad z=\frac{4}{3}x^2y-\frac{1}{3}x^2y^4-2\log y+c \quad ...(15)$$

when $y=1$, the values of z from (8) and (15) should be the same. Hence

$$\frac{4}{3}x^2-\frac{1}{3}x^2-2\log 1+c=x^2-1 \Rightarrow c=-1.$$

Hence the required surface is given by

$$z=\frac{4}{3}x^2y-\frac{1}{3}x^2y^4-2\log y-1.$$

Again from the last two members, we have

$$\frac{dy}{x}=\frac{dz}{f(A)} \Rightarrow dz=f(A)\frac{dy}{\sqrt{(y^2+A)}}.$$

Hence $z=f(A)\log\left[y+\sqrt{A+y^2}\right]+A'$

$$\Rightarrow \quad z=f\left(x^2-y^2\right)\log(y+x)+F\left(x^2-y^2\right).$$

Example 10: Solve $q(1+q)r-(p+q+2pq)s+p(1+p)t=0$.

Solution: Putting $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$ in the given equation, we have

$$q(1+q)\frac{dp-s\,dy}{dx}-(p+q+2qp)s+p(1+p)\frac{dq-s\,dx}{dy}=0$$

$$\Rightarrow \left\{q(1+q)\,dp\,dy+p(1+p)\,dq\,dx\right\}$$
$$-s\left\{q(1+q)\,dy^2+(p+q+2pq)\,dx\,dy+p(1+p)\,dx^2\right\}=0.$$

Hence Monge's subsidiary equations are

$$q(1+q)\,dp\,dy+p(1+p)\,dq\,dx=0 \qquad ...(1)$$

and $q(1+q)\,dy^2+(p+q+2qp)\,dx\,dy+p(1+p)\,dx=0$ ...(2)

Equation (2) gives $p\,dx+q\,dy=0$ ...(3)

and $(1+p)\,dx+(1+q)\,dy=0$ ...(4)

From (3), we have $dz=p\,dx+q\,dy=0 \Rightarrow z=A$.
Also from (1) and (3), we have

$$p(1+q)\,dp-p(1+q)\,dq=0 \Rightarrow \frac{dp}{1+p}-\frac{dq}{1+q}=0.$$

Therefore $\log(1+p)-\log(1+q)=\log B \Rightarrow (1+p)=(1+q)f_1(z)$ ...(5)

which is one intermediate integral.
Now from (4), we have $dx+dy+(p\,dx+q\,dy)=0$

$$\Rightarrow \quad dx+dy+dz=0 \Rightarrow x+y+z=A.$$

From (1) and (4), we have $q\,dp-p\,dq=0 \Rightarrow \dfrac{dp}{p}=\dfrac{dq}{q}$

$$\Rightarrow \quad \log p=\log q+\log B$$

Hence $p=q\,f_2(x+y+z)$. ...(6)

Solving (5) and (6), we have $p=\dfrac{(f_1-1)f_2}{(f_2-f_1)}$ and $q=\dfrac{(f_1-1)}{(f_2-f_1)}$.