It can be expressed as:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + b^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

vii. Pascal's Triangle of $\binom{n}{k}$

Index of binomial	Coefficient of various terms
$(a+b)^0 \rightarrow$	1
$(a+b)^1 \rightarrow$	1 1
$(a+b)^2 \rightarrow$	1 2 1
$(a+b)^3 \rightarrow$	1 3 3 1
$(a+b)^4 \rightarrow$	1 4 6 4 1

• •

viii. Binomial Sums

$$\binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \dots + \binom{k}{k} = \binom{n+1}{k+1}$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n \binom{n}{n} = 0$$

ix. Special Cases

$$(1 \pm x)^n = 1 \pm \binom{n}{1} x + \binom{n}{2} x^2 \pm \dots$$

If n is a positive integer, the series is finite.

$$\left(1 \pm \frac{1}{n}\right)^n = 1 \pm \binom{n}{l} n^{-1} + \binom{n}{2} n^{-2} \pm \dots$$

If n is a negative integer or fraction, the series is infinite.

If $n \rightarrow \infty$, then $\left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = e = 2.718281\dots$ (Euler's number).

x. Progression

An arithmetic progression is a sequence in which the difference between only term and the proceeding term is a constant (d).

$$a, a+d, a+2d, \dots, a+(n-1)d$$

This is called *Riemann's theorem*.

vi. Finite Fourier Transforms

The *finite Fourier sine transform* of $F(x)$, $0 < x < l$, is defined as

$$f_s(n) = \int_0^1 F(x) \sin \frac{n\pi x}{l}\, dx$$

where n is an integer. The function $F(x)$ is then called the *inverse finite Fourier sine transform* of $f_s(n)$ and is given by

$$F(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi x}{l}$$

The *finite Fourier cosine transform* of $F(x)$, $0 < x < l$, is defined as

$$f_s(n) = \int_0^1 F(x) \cos \frac{n\pi x}{l}\, dx$$

where n is an integer. The function $F(x)$ is then called the *inverse finite Fourier cosine transform* of $f_c(n)$ and is given by

$$F(x) = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \cos \frac{n\pi x}{l}$$

vii. The Fourier Integral

Let $F(x)$ satisfy the following conditions:

1. $F(x)$ satisfies the Dirichlet conditions in every finite interval $-l \le x \le l$.
2. $\int_{-\infty}^{\infty} |F(x)|\, dx$ converges, *i.e.* $F(x)$ is absolutely integrable in $-\infty < x < \infty$.

Then *Fourier's integral theorem* states that

$$\boldsymbol{F(x) = \int_0^{\infty} \{A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x\}\, d\lambda}$$

where

$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(x) \cos \lambda x\, dx$$

$$B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(x) \sin \lambda x\, dx$$

This can be written equivalently as

$$F(x) = \frac{1}{2\lambda} \int_{\infty=-\infty}^{\infty} \int_{u=-\infty}^{\infty} F(u) \cos \lambda (x-u) du\, d\lambda$$

iii. Newton's Formula, Backward Interpolation

$$\bar{y}(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots + \frac{u(u+1)(u+2)\ldots(u+n-1)}{n!}\nabla^n y_n \qquad u = \frac{x-x_n}{h}$$

iv. Stirling's Interpolation Formula*

$$\bar{y}(x) = y_0 + u\frac{\Delta y_0 + \nabla y_0}{2} + \frac{u^2}{2!}\Delta\nabla y_0 + \frac{u(u^2-1)}{3!}\frac{\Delta^2\nabla y_0 + \Delta\nabla^2 y_0}{2} + \frac{u^2(u^2-1)}{4!}\Delta^2\nabla^2 y_0 + \ldots \qquad u = \frac{x-x_0}{h}$$

v. Bessel's Interpolation Formula* ($v \neq 0$)

$$\bar{y}(x) = \frac{y_0 + y_1}{2} + v\,\Delta y_0 + \frac{v^2 - \frac{1}{4}}{2}\,\frac{\Delta\nabla y_0 + \Delta\nabla y_1}{2} + \frac{v\left(v^2 - \frac{1}{4}\right)}{3!}\Delta^2\nabla y_0 + \frac{\left(v^2-\frac{1}{4}\right)\left(v^2-\frac{9}{4}\right)}{4!}\,\frac{\Delta^2\nabla^2 y_0 + \Delta^2\nabla^2 y_1}{2} + \ldots \qquad v = u - \frac{1}{2} = \frac{x-x_0}{h} - \frac{1}{2}$$

vi. Bessel's Interpolation Formula* ($v = 0$)

$$\bar{y}(x) = \frac{1}{2}\left[(y_0 + y_1) - \frac{1}{8}(\Delta\nabla y_0 + \Delta\nabla y_1) + \frac{3}{128}\left(\Delta^2\nabla^2 y_0 + \Delta^2\nabla^2 y_1\right) - \frac{5}{1{,}024}\left(\Delta^3\nabla^3 y_0 + \Delta^3\nabla^3 y_1\right) + \ldots\right]$$

$$\nabla y_i = \Delta y_{i-1},\ \Delta\nabla y_j = \Delta^2 y_{j-2},\ \Delta^p\nabla^q y_j = \Delta^{p+q} y_{j-q}$$

18.9 NUMERICAL INTEGRATION, DIFFERENCE POLYNOMIALS

i. Concept

Whenever the closed-form integration becomes too involved or is not feasible, the numerical value of a definite integral can be found (to any degree of accuracy) by means of any of several *quadrature formulas* which express the given integral as a linear combination of a selected set of integrands. The basis of these formulas are *difference polynomials or orthogonal polynomials*.

ii. Trapezoidal Rule [*n* even or odd; *h* = (*b* – *a*)/*n*]

$$\int_a^b y(x)\,dx = \int_a^b \bar{y}(x)\,dx = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-2} + 2y_{n-1} + y_n) + \varepsilon_T$$

Truncation error: $\varepsilon_T \approx -\dfrac{n(h)^3 f''(\xi)}{12}$ $\qquad a \le \xi \le b$

* $\nabla y_j = \Delta y_{j-1},\ \Delta\nabla y_j = \Delta^2 y_{j-2},\ \Delta^p\nabla^q y_j = \Delta^{p+q} y_{j-q}$.

$$\int \frac{\sqrt{C^3}}{x}\,dx = \frac{1}{3}\sqrt{C^3} + a^2\sqrt{C} - a^3 \ln \frac{a+\sqrt{C}}{x}$$

$$\int \frac{\sqrt{C^n}}{x}\,dx = \frac{1}{n}\sqrt{C^n} + a^2 \int \frac{\sqrt{C^{n-2}}}{x}\,dx$$

$$\int \frac{\sqrt{C}}{x^2}\,dx = -\frac{1}{x}\sqrt{C} + \ln(x+\sqrt{C})$$

$$\int \frac{\sqrt{C^3}}{x^2}\,dx = -\frac{1}{x}\sqrt{C^3} + \frac{3x}{2}\sqrt{C} + \frac{3a^2}{2}\ln(x+\sqrt{C})$$

$$\int \frac{\sqrt{C^n}}{x^2}\,dx = \frac{\sqrt{C^n}}{(n-1)\,x} + \frac{a^2 n}{n-1}\int \frac{\sqrt{C^{n-2}}}{x^2}\,dx \qquad n \neq 1$$

$$\int \frac{\sqrt{C}}{x^3}\,dx = -\frac{1}{2x^2}\sqrt{C} - \frac{1}{2a}\ln \frac{a+\sqrt{C}}{x}$$

$$\int \frac{\sqrt{C^3}}{x^3}\,dx = -\frac{1}{2x^2}\sqrt{C^3} + \frac{3}{2}\sqrt{C} - \frac{3a}{2}\ln \frac{a+\sqrt{C}}{x}$$

$$\int \frac{\sqrt{C^n}}{x^3}\,dx = \frac{\sqrt{C^n}}{(n-2)x^2} + \frac{a^2 n}{n-2}\int \frac{\sqrt{C^{n-2}}}{x^3}\,dx \qquad n \neq 2$$

$$\int \frac{\sqrt{C}}{x^4}\,dx = -\frac{\sqrt{C^3}}{3a^2x^3}$$

$$\int \frac{\sqrt{C^3}}{x^4}\,dx = -\frac{\sqrt{C^3}}{3x^3} - \frac{\sqrt{C}}{x} + \ln(x+\sqrt{C})$$

$$\int \frac{\sqrt[p]{C^n}}{x^m}\,dx = \frac{p\sqrt[p]{C^n}}{(2n-mp+p)\,x^{m-1}} + \frac{2a^2 n}{2n-mp+p}\int \frac{\sqrt[p]{C^{n-p}}}{x^m}\,dx \qquad 2n = -p\,(1-m)$$

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xxviii. Indefinite Integrals Involving $f(x) = x^m \sqrt[p]{E^n}$ $\quad E = a^2 - x^2 \quad a^2 \neq 0$

$$\int \sqrt{E}\,dx = \frac{1}{2}\left(x\sqrt{E} + a^2 \sin^{-1}\frac{x}{a}\right)$$

$$\int \sqrt{E^3}\,dx = \frac{1}{8}\left(2x\sqrt{E^3} + 3a^2 x\sqrt{E} + 3a^4 \sin^{-1}\frac{x}{a}\right)$$