of 'back substitution'. The elimination procedure is started from the equation which has the largest coefficient for the first unknown x_1. The value of x_1 is expressed in terms of the other unknowns, i.e. $x_2, x_3, ...,$ etc. is substituted in other equations. The elimination procedure is repeated until only one unknown remains in the last equation and its value determined. The other unknowns are evaluated by back substitution. The method is illustrated through the following practical examples.

Example 1.5

Solve the following set of equations by Gaussian elimination method.

$$3x_1 + x_2 + x_3 = 4 \tag{1}$$
$$x_1 + 4x_2 + x_3 = -5 \tag{2}$$
$$x_1 + x_2 - 6x_3 = -12 \tag{3}$$

Solution

Step 1

Consider Eq. (1) having the largest coefficient of x_1 and divide it by 3 and rewrite the equation as

$$x_1 = (1.33 - 0.33x_2 - 0.33x_3) \tag{4}$$

Step 2

Substitute Eq. (4) in Eqs. (2) and (3)

$$(1.33 - 0.33x_2 - 0.33x_3) + 4x_2 - x_3 = -5$$
$$3.67x_2 - 1.33x_3 = -6.33 \tag{5}$$
$$(1.33 - 0.33x_2 - 0.33x_3) + x_2 - 6x_3 = -12$$
$$0.67x_2 - 6.33x_3 = -13.33 \tag{6}$$

Step 3

Consider Eq. (5) has the largest coefficient of

$$3.67x_2 - 1.33x_3 = -6.33$$
$$3.67x_2 = -6.33 + 1.033x_3 \tag{7}$$
$$\overline{x_2 = -1.72 + 0.362x_3}$$

Step 4

Substitute Eq. (7) in Eq. (6)

$$0.67(-1.72 + 0.362x_3) - 6.33x_3 = -13.33$$

Therefore $x_3 = 2$

Step 5

By back substitution, evaluate x_1 and x_2

From Eq. (7), $x_2 = -1.72 + 0.362(2) = -1$

From Eq. (4), $x_1 = 1.33 + 0.33(-1) - 0.33(2) = 1$

Therefore $x_1 = 1, \quad x_2 = -1, \quad x_3 = 2$

Example 1.6

Solve the following set of equations by the Gaussian elimination method.

Solution

$$w + x + y = 3 \tag{1}$$
$$-3w - 17x + y + 2z = 1 \tag{2}$$
$$4w + 8y - 5z = 1 \tag{3}$$
$$-5x - 2y + z = 1 \tag{4}$$

where L is the unit lower triangular matrix and U is the upper triangular matrix. The triangular matrices are represented as

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 0 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = [A]$$

The matrix representation shown above is termed as factorisation. The diagonal non-zero entry of L matrix is the ith pivot of factorisation. Multiplying L and U, we can evaluate all the elements of $[A]$. The factorisation method is generally preferred over the Gaussian elimination method in computers having the facility of accumulating products in double length. The application of this method is illustrated by the following examples.

Example 1.9

Solve the following system of linear equations using factorisation method.

$$5x - 2y + z = 4 \tag{1}$$

$$7x + y - 5z = 8 \tag{2}$$

$$3x + 7y + 4z = 10 \tag{3}$$

Solution

The equation can be expressed in the matrix following form as,

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 8 \\ 10 \end{Bmatrix}$$

The coefficient matrix is written in the following form.

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

$$[L][U] = [A]$$

The elements of the triangular matrix are determined by multiplying $[L]$ and $[U]$ and equating to the corresponding elements in the coefficient matrix as outlined in the following steps.

$$1(U_{11}) + 0(0) + 0 = 5$$

$$\therefore U_{11} = 5 \tag{4}$$

$$1(U_{12}) + 0(U_{22}) + 0 = -2$$

$$\therefore U_{12} = -2 \tag{5}$$

$$1(U_{13}) + 0(U_{23}) + 0(U_{33}) = 1$$

$$\therefore U_{13} = 1 \tag{6}$$

$$g_{31} = \left(\frac{a_{13}}{g_{11}}\right) = \left(\frac{-1}{1.732}\right) = -0.577$$

$$g_{22} = \sqrt{a_{22} - g_{21}^2} = \sqrt{4 - (1.555)^2} = 1.633$$

$$g_{32} = \left(\frac{a_{23} - g_{21}g_{31}}{g_{22}}\right) = \left[\frac{2 - (1.155)(-0.577)}{1.633}\right] = 1.633$$

$$g_{33} = \sqrt{a_{33} - g_{31}^2 - g_{32}^2} = \sqrt{4 - [(-0.577)^2 + 1.633^2} = 1.0$$

The resulting matrix is

$$[G] = \begin{bmatrix} 1.732 & 0 & 0 \\ 1.155 & 1.633 & 0 \\ -0.577 & 1.633 & 1 \end{bmatrix}$$

Step 2. Forward solution

$$[G][Y] = [C]$$

$$\begin{bmatrix} 1.732 & 0 & 0 \\ 1.155 & 1.633 & 0 \\ -0.577 & 1.633 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}$$

Solving the first equation for y_1 gives

$$1.732y_1 = 4$$
$$\therefore y_1 = 2.309$$

Solving the second equation for y_2 gives

$$1.155y_1 + 1.633y_2 = 8$$

$$1.155(2.309) + 1.633y_2 = 8 \qquad \therefore y_2 = 3.266$$

Solving the third equation for y_3 gives

$$-0.577y_1 + 1.633y_2 + y_3 = 5$$

$$-0.577(2.309) + 1.633(3.266) + y_3 = 5$$

Solving $y_3 = 1.000$

Step 3. Backward solution

Solve the equation

$$[G]^T [X] = [Y]$$

$$\begin{bmatrix} 1.732 & 1.155 & -0.577 \\ 0 & 1.633 & 1.633 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2.309 \\ 3.266 \\ 1.000 \end{bmatrix}$$

Solution

The equations are expressed in the matrix form

$$\begin{bmatrix} 1 & 2 & -1 \\ 4 & -3 & 4 \\ 2 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} -3 \\ 1 \\ -2 \end{Bmatrix}$$

$$[A]\{X\} = \{K\}$$

Therefore $X = [A^{(-1)}]K$, where A^{-1} is the inverse of matrix A.

The cofactors of the matrix A are determined as follows:

$$A_{11} = (-1)^2 \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -3 + 4 = 1$$

$$A_{12} = (-1)^3 \begin{vmatrix} 4 & 4 \\ 2 & 1 \end{vmatrix} = (-1)(4 - 8) = 4$$

$$A_{13} = (-1)^4 \begin{vmatrix} 4 & -3 \\ 2 & -1 \end{vmatrix} = -4 + 6 = 2$$

$$A_{21} = (-1)^3 \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = (-1)(2 - 1) = -1$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1 + 2 = 3$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = (-1)(-1 - 4) = 5$$

$$A_{31} = (-1)^4 \begin{vmatrix} 2 & -1 \\ -3 & 4 \end{vmatrix} = 8 - 3 = 5$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 4 & 4 \end{vmatrix} = (-1)(4 + 4) = -8$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & -2 \\ 4 & -3 \end{vmatrix} = -3 - 8 = -11$$

Therefore $\text{adj.}\, A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

$$= \begin{bmatrix} 1 & 4 & 2 \\ -1 & 3 & 5 \\ 5 & -8 & -11 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 5 \\ 4 & 3 & -8 \\ 2 & 5 & -11 \end{bmatrix}$$